ANALYSIS QUALIFYING EXAM

JUNE 2012

REAL ANALYSIS

Answer all 4 questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Question 1 (30 points)
a) Let $f_n : X \to \mathbb{R}$ be a sequence of $(X, \mathcal{M})$ measurable functions. Prove that the set of points $E = \{x \mid \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.
b) Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$ and define $\nu = \mu|_{\mathcal{N}}$. If $f \in L^1(\mu)$ prove there exists a $g \in L^1(\nu)$ such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathcal{N}$. Also show that g is defined uniquely ν a.e.

Question 2 (20 points)
Let μ, ν be finite measure on (X, M) with $\nu << \mu$. Define $\lambda = \mu + \nu$ and $f = \frac{d\nu}{d\lambda}$. Prove that $0 \le f < 1$ $\mu-$a.e. and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

Question 3 (30 points)
a) Let E be a Borel set in $\mathbb{R}^n$ and m denote Lebesgue measure on $\mathbb{R}^n$. Let

$$D_E(x) = \lim_{r\to 0} \frac{m(E \cap B(r, x))}{m(B(r, x))}$$

whenever the limit exists. Prove that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.
b) Prove that Haar measure for a compact group or abelian group is both left and right invariant.

Question 4 (20 points)
Let X, Y be Banach spaces. If $T : X \to Y$ is a linear map such that $f \circ T \in X^*$ for all $f \in Y^*$ then T is bounded.

COMPLEX ANALYSIS

You should attempt all the problems. Partial credit will be give for serious efforts.
Each problem is worth 25 points

(1) Compute the following integral

$$\int_0^\infty \frac{\cos x}{1+x^4}\,dx$$

(2) Prove that the function

$$f(z) = \sum z^{n!}$$

cannot be analytically continued to any open set strictly larger than the unit disk.

(3) Prove that a one-to-one entire function must be a linear function.

(4) Let Ω be a simply connected open set in $\mathbb{C}$ and suppose $\Omega \neq \mathbb{C}$. Let $f : \Omega \to \Omega$ be a holomorphic mapping. Prove that $f(z)$ cannot have more than one fixed point unless $f(z) = z$ for all $z \in \Omega$. (A point α is called a fixed point if $f(\alpha) = \alpha$.)