Algebra Qualifying Exam
Fall 2015
3 hours

1. Classify groups of order 55 up to isomorphism. Give a presentation for each of the groups in your classification.

2. Let $R = \mathbb{C}[X, Y]$ and consider the ideal $I = (X, Y)$ as an R-module.

(a) Construct an exact sequence of R-modules

$$0 \to R \to R \oplus R \to I \to 0.$$

(b) Prove that the sequence you constructed is not split.

3. Consider the ideal

$$I = (X^2 - Y, Y^2 - X) \subset \mathbb{C}[X, Y].$$

Find all maximal ideals of the quotient $\mathbb{C}[X, Y]/I$. (*Find* means give a set of generators.)

4. How many Sylow p-subgroups are there in $\mathrm{GL}_2(\mathbb{F}_p)$?

5. Suppose K is an extension of $\mathbb{Q}$ of degree n, and let $\sigma_1, \ldots, \sigma_n : K \to \mathbb{C}$ be the distinct embeddings of K into $\mathbb{C}$. Let $\alpha \in K$. Regarding K as a $\mathbb{Q}$-vector space, let $\phi : K \to K$ be the linear transformation $\phi(x) = \alpha x$. Show that the eigenvalues of ϕ are $\sigma_1(\alpha), \ldots, \sigma_n(\alpha)$.

6. Let $\zeta = e^{\pi i/3} \in \mathbb{C}$.

(a) Compute the minimal polynomial of ζ over $\mathbb{Q}$.

(b) Find all prime ideals $\mathfrak{p} \subset \mathbb{Z}[\zeta]$ satisfying $\mathfrak{p} \cap \mathbb{Z} = 7\mathbb{Z}$, and give generators for these ideals.

7. Fix $a, b, c \in \mathbb{Q}$, and let $K/\mathbb{Q}$ be the splitting field of

$$f(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 \in \mathbb{Q}[x].$$

Show that $\mathrm{Gal}(K/\mathbb{Q}) \subset S_6$ is contained in the centralizer of a permutation with cycle type $(2, 2, 2)$.

8. Let R be a ring with identity, and let M be a left R-module. Prove that the following are equivalent:

(i) there is a chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n = (0)$ such that each quotient M_j/M_{j+1} is a simple R- module (*simple* means no proper nonzero submodules)

(ii) M satisfies both the ascending chain condition and the descending chain condition for submodules.

Hint: For (i) $\implies$ (ii), use induction on n.