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RECURSIVE AMBIGUITY AND MACHINA'S EXAMPLES*

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Machina (*American Economic Review* 99 (2009), 385–392; *American Economic Review 104* (2014), 3814–40) lists a number of situations where Choquet expected utility, as well as other known models of ambiguity aversion, cannot capture plausible features of ambiguity attitudes. Most of these problems arise in choice over prospects involving three or more outcomes. We show that the recursive nonexpected utility model of Segal (*International Economic Review* 28 (1987), 175–202) is rich enough to accommodate all these situations and, moreover, that this can be done using the same functional form for all situations.

1. INTRODUCTION

Ambiguity aversion is one of the most investigated phenomenon in decision theory. Ambiguity refers to situations where a decision maker does not know the exact probabilities of some events. The claim that decision makers systematically prefer betting on events with known instead of with unknown probabilities, a phenomenon known as ambiguity aversion, was first suggested in a series of examples by Ellsberg (1961) and was soon proved to hold true in many experiments. The importance of Ellsberg's findings stems from the fact that they cannot be reconciled with individuals holding *any* subjective probabilities over events. Mainly motivated by Ellsberg's examples, several formal models have been proposed to accommodate ambiguity aversion. One of the most important models in the literature, known as Choquet expected utility (Schmeidler, 1989), assumes that decision makers hold nonadditive beliefs (called capacities), which overweight events associated with bad outcomes.

Ellsberg's experiments involve binary bets (that is, the ambiguous prospects have only two possible outcomes). Machina (2009) claims that there are some aspects of ambiguity aversion that arise only in the presence of nonbinary bets. For example, if there are three possible monetary outcomes $a > b > c$, then a decision maker may prefer ambiguity about the probabilities of receiving a and b to ambiguity about the probabilities of receiving b and c. Accordingly, Machina (2009) suggests some examples that involve three or more outcomes and shows that plausible attitudes toward ambiguity in these problems cannot be accommodated by Choquet expected utility. Baillon et al. (2011) show that Machina's examples pose difficulties not only for Choquet expected utility but for several other known models as well.[2,3] In a follow-up paper, Machina (2014) offers more thought experiments of nonbinary bets and explains why they pose new difficulties for Choquet expected utility as well as to some other models.

Machina's examples are in line with a well-established tradition of "puzzles" in decision theory: A theory implies a specific relationship between two choice problems, even though

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[2] Maxmin expected utility (Gilboa and Schmeidler, 1989), variational preferences (Maccheroni et al., 2006), α-maxmin (Ghirardato et al., 2004), and the smooth model of ambiguity aversion (Klibanoff et al., 2005).

[3] Baillon et al. (2011) give an example of general preferences that are consistent with the two examples of Machina (2009). As they point out, this example is not particularly intuitive. Similarly to the functional we use in this article, their example does not feature expected utility on a purely objective domain (lotteries). Baillon et al. also mention that some version of Siniscalchi's (2009) vector-expected utility is able to account for the same two examples.

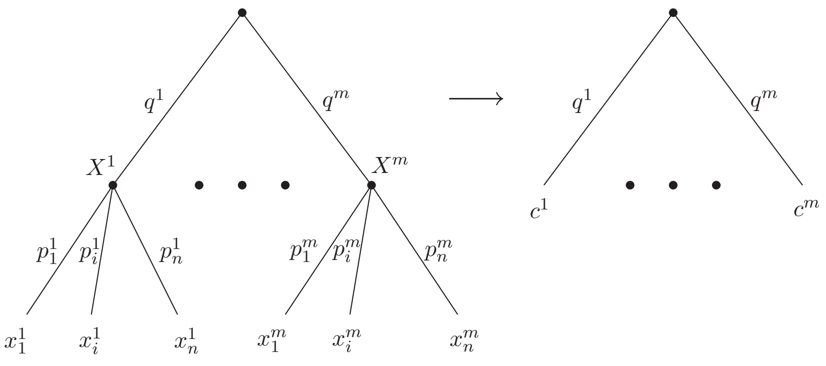

FIGURE 1

RECURSIVE EVALUATION OF A TWO-STAGE LOTTERY

thought or actual experiments systematically violate this relationship. Such are, for instance, the aforementioned Ellsberg's examples that challenge the subjective expected utility model of Savage (1954) and, in the context of decision making under risk, Allais (1953) paradox. In a similar way, Machina's examples challenge the links between different decision situations implied by Choquet expected utility.

In this article, we show that all of Machina's examples can be handled by the two-stage recursive ambiguity model of Segal (1987) and, moreover, that this can be done using the same functional form for all examples. According to the recursive model, ambiguity corresponds to the case where there is some set of states of the world and the decision maker does not know the exact probability distribution over these states. Instead, he has in mind a set of conceivable distributions and, furthermore, he is able to assign (subjective) probabilities to the different distributions in this set. For each distribution, the decision maker computes its certainty equivalent using some nonexpected utility functional. He then views the uncertain prospect as a lottery over these certainty equivalents and evaluates it using the same nonexpected utility functional. We provide some simple examples demonstrating that the recursive model is rich enough not to impose the links between different decision situations that exist in Choquet expected utility. While without further restrictions the recursive model is very general, we show that a single functional form can address all the aspects described in Machina's examples.

The reminder of the article is organized as follows: Section 2 reviews the recursive nonexpected utility model. Section 3 describes Machina's examples and shows how they can be accommodated by the recursive model.

2. RECURSIVE NONEXPECTED UTILITY

In this section, we outline the recursive nonexpected utility model of Segal (1987) and the special case of it we invoke in our analysis. Let $[w, b]$ be an interval of monetary prizes, and let $S = \{s_1, \ldots, s_n\}$ be a finite state space. Consider an act $x = (x_1, s_1; \ldots; x_n, s_n)$, which pays the amount x_i if state s_i happens. The decision maker does not know the probabilities of the states $s_1, \ldots, s_n$, but he has in mind a set of possible probability measures over them. For simplicity, assume that there are m such possible measures, $P^j = (p_1^j, \ldots, p_n^j)$, $j = 1, \ldots, m$ (here p_i^j is the probability that state s_i occurs under the measure P^j). The decision maker holds subjective beliefs about the likelihood of each measure in the set. In particular, he believes that with probability q^j the true measure is P^j. He therefore views the ambiguous prospect as a two-stage lottery $(X^1, q^1; \ldots; X^m, q^m)$, where with probability q^j he will play the single-stage lottery $X^j = (x_1, p_1^j; \ldots; x_n, p_n^j)$, $j = 1, \ldots, m$. This two-stage lottery is depicted on the left-hand side of Figure 1.

The decision maker is using a nonexpected utility functional V to evaluate single-stage lotteries. Denote by c^j the certainty equivalent of lottery X^j, that is, the number that satisfies

$$V(c^j, 1) = V\left(x_1, p_1^j; \ldots; x_n, p_n^j\right).$$

The decision maker evaluates acts using the following two-step approach: He first replaces each of the m second-stage lotteries $X^1, \ldots, X^m$ with its certainty equivalent calculated using the functional V, thus obtaining the simple lottery $(c^1, q^1; \ldots; c^m, q^m)$, as seen on the right-hand side of Figure 1. He then computes the V value of this lottery, $V(c^1, q^1; \ldots; c^m, q^m)$, which is his subjective value of the ambiguous act x.

The decision maker may instead reduce the two-stage lottery into a simple lottery by computing the overall probabilities of the states, that is, he may identify the act x with the lottery that pays x_i with probability $\sum_{j=1}^{m} q^j p_i^j$. This is known as the reduction of compound lotteries axiom, and together with the above recursive procedure is known to imply, and be implied by, expected utility theory (see Samuelson, 1952, and Segal, 1990). The procedure we use must therefore violate the reduction axiom and expected utility theory. For further analysis, see Segal (1987, 1990).[4] This has two important consequences. First, experimental evidence throughout the years emphasizes the descriptive limitations of expected utility. Our model is consistent with more general behavioral patterns under risk (e.g., the Allais paradox). Second, Machina (2009) shows that many of his examples are driven by some event-separability properties of Choquet expected utility. For instance, if two acts pay the same on some event E and if the payoff on E affects the value of each act independently of its payoffs on other events, then the comparison of these two acts should not depend on the exact magnitude of the payoff on E, as long as it is the same in both. But changes of the payoffs on E may change the ambiguity properties of the two acts (e.g., transform any of the acts from being fully objective to subjective, or vice versa), causing an ambiguity averse decision maker to alter their ranking. Consequentially, Machina (2009) argues that "nonseparable models of preferences might be better at capturing features of behavior that lead to these paradoxes." On the other hand, the recursive model with nonexpected utility implies a lot of nonseparability between the outcomes. The evaluation of each of the lotteries $X^1, \ldots, X^m$ without expected utility implies interdependency between outcomes, and even if partial separability exists, it typically disappears when the lottery over the certainty equivalents $c^1, \ldots, c^m$ is evaluated.

Identifying ambiguity with a compound lottery that the decision maker fails to reduce does not depend on the specific functional V used in the evaluation procedure described above. But since we would like to show that all Machina's examples can be accommodated by the same functional form, in this article we confine our attention to a specific nonexpected utility functional, namely, Gul's (1991) model of disappointment aversion. The disappointment aversion value of the single-stage lottery $X = (x_1, p_1; \ldots; x_n, p_n)$ is the unique v that solves

$$v = \frac{\sum_{\{x_i : u(x_i) \geqslant v\}} p_i u(x_i) + (1 + \beta) \sum_{\{x_i : u(x_i) < v\}} p_i u(x_i)}{1 + \beta \sum_{\{x_i : u(x_i) < v\}} p_i}, \tag{1}$$

where $\beta \in (-1, \infty)$ and $u : [w, b] \to \Re$ is increasing. In the disappointment aversion model, the support of any nondegenerate lottery is divided into two groups, the elating outcomes (which are preferred to the lottery) and the disappointing outcomes (which are worse than the lottery). The decision maker evaluates lotteries by taking their "expected utility," except that disappointing outcomes get a uniformly greater (or smaller) weight that depends on the value of a single parameter β, the coefficient of disappointment aversion. Throughout the article, we further assume linear utility over outcomes, $u(x) = x$, and $\beta = 0.2$.

[4] Halevy (2007) provides evidence in favor of the recursive, nonexpected utility model. Approximately 40% of his subjects were classified as having preferences that are consistent with that model.

TABLE 1
THE 50:51 EXAMPLE

	50 Balls		51 Balls	
Act	E_1	E_2	E_3	E_4
f_1	8,000	8,000	4,000	4,000
f_2	8,000	4,000	8,000	4,000
f_3	12,000	8,000	4,000	0
f_4	12,000	4,000	8,000	0

Under the interpretation that ambiguity aversion amounts to preferring the objective (unambiguous) simple lottery to the (ambiguous) compound one, Artstein-Avidan and Dillenberger (2011) show that a disappointment averse decision maker with $\beta > 0$ displays ambiguity aversion for *any* possible beliefs he might hold about the probability distribution over the states.[5] Therefore, this functional is consistent with Ellsberg's examples. And as we show in this article, it is also consistent with all of Machina's examples.

3. ADDRESSING MACHINA'S EXAMPLES

The first two examples are taken from Machina (2009). The other examples are taken from Machina (2014). For each example, we state the decision maker's beliefs (and the two-stage lotteries they induce). All rankings are based on applying the recursive model using the disappointment aversion functional V with $u(x) = x$ and $\beta = 0.2$.

3.1. *The 50:51 Example.* An urn contains 101 balls, each carries one of the numbers $1, \ldots, 4$. Of these, 50 are marked either 1 or 2 and 51 are marked either 3 or 4. Let E_i denote the event "a ball marked i is drawn" and consider the four acts shown in Table 1.

Machina shows that Choquet expected utility implies that $f_1 \succeq f_2$ if and only if $f_3 \succeq f_4$. Nevertheless, Machina (2009, section II) invokes an Ellsberg-like argument that f_4 could be preferred to f_3 even though f_1 were preferred to f_2, which accordingly violates Choquet expected utility theory.

We now analyze the four acts $f_1, \ldots, f_4$ using the recursive model. Suppose that the decision maker believes that 25 balls are marked 1 and 25 balls are marked 2. With respect to the composition of the other 51 balls, he believes that it is equally likely that either all of them are marked 3 or all of them are marked 4.[6] The acts $f_1, \ldots, f_4$ induce the following two-stage lotteries (to simplify notation, we divide all outcomes by 1,000):

$$f_1 \rightarrow \left(8, \frac{50}{101}; 4, \frac{51}{101}\right)$$

$$f_2 \rightarrow \left(\left(8, \frac{76}{101}; 4, \frac{25}{101}\right), \frac{1}{2}; \left(8, \frac{25}{101}; 4, \frac{76}{101}\right), \frac{1}{2}\right)$$

$$f_3 \rightarrow \left(\left(12, \frac{25}{101}; 8, \frac{25}{101}; 4, \frac{51}{101}\right), \frac{1}{2}; \left(12, \frac{25}{101}; 8, \frac{25}{101}; 0, \frac{51}{101}\right), \frac{1}{2}\right)$$

$$f_4 \rightarrow \left(\left(12, \frac{25}{101}; 8, \frac{51}{101}; 4, \frac{25}{101}\right), \frac{1}{2}; \left(12, \frac{25}{101}; 4, \frac{25}{101}; 0, \frac{51}{101}\right), \frac{1}{2}\right)$$

[5] This assertion is not specific to Gul's model but applies to any member of the class of preferences characterized in Dillenberger (2010) and in Cerreia-Vioglio et al. (forthcoming).

[6] This particular choice is not crucial for our result. That is, the argument could be made with many other possible compositions of the urn. The recursive nonexpected utility model does not pin down the beliefs of the decision maker. Our aim is thus to make our point using simple and plausible possible beliefs.

TABLE 2
THE REFLECTION EXAMPLE

	50 Balls		50 Balls	
Act	E_1	E_2	E_3	E_4
f_5	4,000	8,000	4,000	0
f_6	4,000	4,000	8,000	0
f_7	0	8,000	4,000	4,000
f_8	0	4,000	8,000	4,000

TABLE 3
THE SLIGHTLY BENT COIN PROBLEM

I	Black	White	*II*	Black	White
Heads	8,000	0	Heads	0	0
Tails	−8,000	0	Tails	−8,000	8,000

We obtain that $f_1 \succ f_2$ but $f_4 \succ f_3$.

3.2. *The Reflection Example.* Consider the acts shown in Table 2.

The two acts f_5 and f_8 reflect each other, and the decision maker should therefore be indifferent between them. Likewise, f_6 should be indifferent to f_7. As by the Choquet expected utility model $f_5 \succeq f_6$ iff $f_7 \succeq f_8$, it follows that $f_5 \sim f_6$ (and $f_7 \sim f_8$). Yet, as is argued by Machina (2009, section III), ambiguity attitudes may well suggest strict preference within each pair.

Let $\alpha, \beta, \gamma, \delta$ be a list of possible numbers of balls of the four types in the urn, where $\alpha + \beta = \gamma + \delta = 50$. Denote by $q(\alpha, \beta, \gamma, \delta)$ the probability the decision maker attaches to the event "the composition of the urn is $\alpha, \beta, \gamma, \delta$." We say that such beliefs are symmetric if

$$q(\alpha, \beta, \gamma, \delta) = q(\beta, \alpha, \delta, \gamma) = q(\gamma, \delta, \alpha, \beta) = q(\delta, \gamma, \beta, \alpha).$$

If beliefs are symmetric, then the recursive model implies $f_5 \sim f_8$ and $f_6 \sim f_7$, yet it does not require $f_5 \sim f_6$. In fact, it can be shown that such indifference will *not* hold in general. For example, if $q(10, 40, 25, 25) = \frac{1}{4}$ then we have $f_6 \succ f_5$.

3.3. *The Slightly Bent Coin Problem.* A coin is flipped and a ball is drawn out of an urn. You know that the coin is slightly bent (but you do not know which side is more likely or the respective probabilities) and that the urn contains two balls, each is either white or black. Which of the bets given in Table 3 do you prefer?

According to Machina (2014, section IV), it is plausible that an ambiguity averse decision maker will prefer Bets *I* to *II*. The reason is that if the coin is only slightly biased, then betting on the coin flip (as in Bet *I*) is less ambiguous than betting on the color of the ball (as in Bet *II*). Yet he shows that a Choquet expected utility maximizer must be indifferent between the two bets.

Consider first the urn with the two balls. As there is no reason to believe any bias in favor of white or black, we assume that the decision maker believes that the probability of each of the two events "there are two black balls" and "there are two white balls" is q, and the probability of the event "there is one black and one white ball" is $1 - 2q$.

The analysis of the coin is slightly more involved, as the decision maker does not know the direction in which it is biased (heads or tails), nor does he know the magnitude of the bias (that is, the probabilities $p : 1 - p$ of the two sides). For simplicity, we assume that the bias of the coin is equally likely to be either ε or $-\varepsilon$. We thus obtain six possible probability distributions over

TABLE 4
POSSIBLE PROBABILITY DISTRIBUTIONS

Case #	Pr(*head*), # of *black*	Prob.	hb	hw	tb	tw
1	$\frac{1}{2}+\varepsilon, \#b=2$	$\frac{q}{2}$	$\frac{1}{2}+\varepsilon$	0	$\frac{1}{2}-\varepsilon$	0
2	$\frac{1}{2}-\varepsilon, \#b=2$	$\frac{q}{2}$	$\frac{1}{2}-\varepsilon$	0	$\frac{1}{2}+\varepsilon$	0
3	$\frac{1}{2}+\varepsilon, \#b=1$	$\frac{1}{2}-q$	$\frac{1}{4}+\frac{\varepsilon}{2}$	$\frac{1}{4}+\frac{\varepsilon}{2}$	$\frac{1}{4}-\frac{\varepsilon}{2}$	$\frac{1}{4}-\frac{\varepsilon}{2}$
4	$\frac{1}{2}-\varepsilon, \#b=1$	$\frac{1}{2}-q$	$\frac{1}{4}-\frac{\varepsilon}{2}$	$\frac{1}{4}-\frac{\varepsilon}{2}$	$\frac{1}{4}+\frac{\varepsilon}{2}$	$\frac{1}{4}+\frac{\varepsilon}{2}$
5	$\frac{1}{2}+\varepsilon, \#b=0$	$\frac{q}{2}$	0	$\frac{1}{2}+\varepsilon$	0	$\frac{1}{2}-\varepsilon$
6	$\frac{1}{2}-\varepsilon, \#b=0$	$\frac{q}{2}$	0	$\frac{1}{2}-\varepsilon$	0	$\frac{1}{2}+\varepsilon$

TABLE 5
THE UPPER/LOWER TAIL PROBLEM

	Red	Black	White
Urn *I*	100	0	C
Urn *II*	0	C	100

TABLE 6
RECURSIVE ANALYSIS OF THE UPPER/LOWER TAIL PROBLEM

No. of Black Balls	2	1	0
Probability	q	$1-2q$	q
Urn *I*	$(0, \frac{2}{3}; 100, \frac{1}{3})$	$(0, \frac{1}{3}; C, \frac{1}{3}; 100, \frac{1}{3})$	$(C, \frac{2}{3}; 100, \frac{1}{3})$
Urn *II*	$(0, \frac{1}{3}; C, \frac{2}{3})$	$(0, \frac{1}{3}; C, \frac{1}{3}; 100, \frac{1}{3})$	$(0, \frac{1}{3}; 100, \frac{2}{3})$

the four possible events—heads-black (hb), heads-white (hw), tails-black (tb), and tails-white (tw)—depicted in Table 4.

After dividing by 1,000, the payoffs of the two gambles are given by $I = (8, \text{hb}; 0, \text{hw}; -8, \text{tb}; 0, \text{tw})$ and $II = (0, \text{hb}; 0, \text{hw}; -8, \text{tb}; 8, \text{tw})$. If $\varepsilon = 0.05$, and $q = 0.25$ we obtain that $I \succ II$. On the other hand, setting $\varepsilon = 0.25$ and $q = 0.05$ (that is, the coin is seriously biased but the decision maker believes that the two balls are most likely of different color) we obtain that $II \succ I$.

3.4. *The Upper/Lower Tail Problem.* Let C denote your certainty equivalent of the lottery $(100, \frac{1}{2}; 0, \frac{1}{2})$. Urns *I* and *II* contain each one red ball and two other balls, each of which is either white or black. One ball is drawn from an urn of your choice, and the payoffs are given in Table 5. Do you prefer to play urn *I* or *II*?

Machina shows that Choquet expected utility does not allow the decision maker to have strict preferences between these two bets, that is, the model imposes indifference.

Using the analysis of "The slightly bent coin" above, the decision maker believes that the probability of two black balls is q, the probability of two white balls is q, and the probability of one black and one white ball is $1-2q$. The two urns are thus transformed into two-stage lotteries, given by Table 6, and we have $I \succ II$. This breaks the indifference implied by Choquet expected utility, but disagrees with Machina's prediction that an ambiguity averse decision maker should prefer urns *II* to *I*.

4. CONCLUDING REMARKS

Machina (2009, 2014) showed that there are aspects of ambiguity aversion that arise in choice over prospects involving three or more outcomes that cannot be handled by many popular models, including Choquet expected utility. As argued by Machina, the reason is that these models impose too much separability in the way outcomes paid on different events are aggregated in the evaluation procedure. In this article, we show that all these issues can be accommodated by the two-stage recursive ambiguity model of Segal (1987) and, moreover, that this can be done using the same functional form for all examples. In other words, the recursive model, although consistent with the standard intuition of ambiguity aversion with respect to Ellsberg's (1961) examples, is rich enough not to impose connections within Machina's pairs of choices.

The reason Segal's recursive model can handle these examples is that this model can impose no separability between any two outcomes. This nonseparability has two sources. First, each possible distribution over the outcomes is evaluated using a functional V that may impose no separability between the outcomes. But even if V imposes some degree of separability, the lottery over the certainty equivalents (of the possible lotteries) will link some of these values, and as each of the certainty equivalents depends in general on all possible outcomes, nonseparability will emerge. The only case in which this will not happen is when V itself imposes full separability over outcomes. The only functional V to obtain full separability is expected utility, which, as we explained in Section 2, is indeed the only functional that trivializes Segal's recursive model.

REFERENCES

ALLAIS, M., "Le comportement de l'homme rationnel devant le risque: Critique des postulates et axiomes de l'ecole Americaine," *Econometrica* 21 (1953), 503–46.

ARTSTEIN-AVIDAN, S., AND D. DILLENBERGER, "Dynamic Disappointment Aversion," Mimeo, University of Pennsylvania, 2011.

BAILLON A., O. L'HARIDON, AND L. PLACIDO, "Ambiguity Models and the Machina Paradoxes," *American Economic Review* 101(4) (2011), 1547–60.

CERREIA VIOGLIO, S., D. DILLENBERGER, AND P. ORTOLEVA, "Cautious Expected Utility and the Certainty Effect," *Econometrica* (forthcoming).

DILLENBERGER, D., "Preferences for One-Shot Resolution of Uncertainty and Allais-Type Behavior," *Econometrica* 78 (2010), 1973–2004.

ELLSBERG, D., "Risk, Ambiguity, and the Savage Axioms," *Quarterly Journal of Economics* 75 (1961), 643–69.

GHIRARDATO, P., F. MACCHERONI, AND M. MARINACCI, "Differentiating Ambiguity and Ambiguity Attitude," *Journal of Economic Theory* 118 (2004), 133–73.

GILBOA, I., AND D. SCHMEIDLER, "Maxmin Expected Utility with a Non-Unique Prior," *Journal of Mathematical Economics* 18 (1989), 141–53.

GUL, F., "A Theory of Disappointment Aversion," *Econometrica* 59 (1991), 667–86.

HALEVY, Y., "Ellsberg Revisited: An Experimental Study," *Econometrica* 75 (2007), 503–36.

KLIBANOFF, P., M. MARINACCI, AND S. MUKERJI, "A Smooth Model of Decision Making under Ambiguity," *Econometrica* 73 (2005), 1849–92.

MACCHERONI, F., M. MARINACCI, AND A. RUSTICHINI, "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Econometrica* 74 (2006), 1447–98.

MACHINA, M., "Risk, Ambiguity, and the Rank-Dependence Axioms," *American Economic Review* 99 (2009), 385–92.

———, "Ambiguity Aversion with Three or More Outcomes," *American Economic Review* 104, 3814–40.

SAMUELSON, P. A., "Probability, Utility and the Independence Axiom," *Econometrica* 20 (1952), 670–678.

SAVAGE, L. J., *Foundations of Statistics* (New York: John Wiley, 1954).

SCHMEIDLER, D., "Subjective Probability and Expected Utility without Additivity," *Econometrica* 57 (1989), 571–87.

SEGAL, U., "The Ellsberg Paradox and Risk Aversion: An Anticipated Utility Approach," *International Economic Review* 28 (1987), 175–202.

———, "Two-Stage Lotteries without the Reduction Axiom," *Econometrica* 58 (1990), 349–77.

SINISCALCHI, M., "Vector Expected Utility and Attitudes toward Variation," *Econometrica* 77 (2009), 801–55.