Real Analysis Qualifying Exam

Answer all four questions. In your proofs, you may use any major theorem, except the result you are trying to prove (or a variant of it). State clearly what theorems you use. All four questions are worth the same number of points. Good luck.

Notation: In the questions below, λ^n denotes Lebesgue measure on $\mathbb{R}^n$.

Question 1. Let $\{f_1, f_2, \dots\}$ be a sequence of continuous, positive functions defined on the unit interval $[0, 1]$ with

$$\int_0^1 f_n(x)\ d\lambda^1(x) = 1$$

for all n. Assume that the pointwise limit of the sequence $\{f_n\}$ exists, and denote it by f.

a. Is it always true that $\int_0^1 f(x)\ d\lambda^1(x) \leq 1$? Prove or provide a counterexample.

b. Is it always true that $\int_0^1 f(x)\ d\lambda^1(x) \geq 1$? Prove or provide a counterexample.

Question 2. For any λ^2-measurable function $f : \mathbb{R}^2 \to \mathbb{R}$, and for every $x, y \in \mathbb{R}$, define $f_x : \mathbb{R} \to \mathbb{R}$ and $f^y : \mathbb{R} \to \mathbb{R}$ by $f_x(p) = f(x, p)$ and $f^y(p) = f(p, y)$.

a. Given an example of such a function f such that $f_x \in L^1(\mathbb{R})$ for a.e. x and $f^y \in L^1(\mathbb{R})$ for a.e. y but

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_x(y) dy dx \neq \int_{\mathbb{R}} \int_{\mathbb{R}} f^y(x) dx dy \qquad (1)$$

b. What does Fubini's theorem assert about such f (f that satisfy (1))?

c. What does Tonelli's theorem assert about such f (f that satisfy (1))?

Question 3. Prove that a normed vector space is a Banach space if and only if every absolutely convergent series is convergent. As part of your answer, state the definitions of "Banach space," "absolutely convergent" and "convergent."

Question 4. Denote by $\mathcal{A}$ the smallest algebra of subsets of $\mathbb{R}$ that contains all bounded intervals. Denote by $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$. Denote by ${\lambda^1}^*$ the outer measure on the power set $\mathcal{P}(\mathbb{R})$ induced by the premeasure on $\mathcal{A}$ that assigns to any bounded interval its Euclidean length, and to any unbounded interval ∞.

a. Let $E \subset \mathbb{R}$. What does "E is ${\lambda^1}^*$-measurable" (i.e. outer measurable) mean?

b. How is the collection of λ^1-measurable sets related to the collection of ${\lambda^1}^*$-measurable sets?

c. Prove that for any $E \subseteq \mathbb{R}$ and any $\epsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ and ${\lambda^1}^*(A) \leq {\lambda^1}^*(E) + \epsilon$.