Complex Analysis Qualifying Exam, Fall 2021

This part of the Analysis Qualifying exam has four problems, each worth 10 points. Show all your work and explain all your reasoning. You may use any result proved in our course, as long as you state clearly what result you are using (including its hypotheses). However, you may not use a result which is the same as the problem you are being asked to do.

1. Let f be a meromorphic function on $\mathbb{C}$. Suppose there are real numbers M and R, and an integer n such that $|f(z)| \leq M|z|^n$ whenever $|z| > R$ and z is not a pole of f. Prove that f is a rational function.

2. Suppose f is holomorphic on the disk $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Prove the following.

(a) $|f'(0)| \leq 1 - |f(0)|^2$.

(b) $f(z) \neq 0$ if $|z| < |f(0)|$.

3. Let f and g be holomorphic functions on a connected open set U in $\mathbb{C}$. Let $fg : U \to \mathbb{C}$ be the function whose value at $z \in U$ is $f(z)g(z)$. Suppose fg is the zero function on U. Prove that either f or g is the zero function on U.

4. Compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{(1+z^2)^2}\, dz$$

where a is a real constant, $a \geq 0$.