Algebra Qualifying Examination, Fall 2019

Instuctions: This is a 3 hour examination. In the problems below, all rings are commutative with identity unless specified otherwise. This is a closed book exam, also no notes, searching the web, or otherwise consulting external sources. Good luck!

1. Let P be a finite p-group. Show that P is not cyclic if and only if P has a quotient isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

2. Let R be a commutative ring with unity.

   (a) Let S be a non-empty saturated multiplicative set in R, i.e. if $a, b \in R$, then $ab \in S$ if and only if $a, b \in S$. Show that $R - S$ (the complement of S in R) is a union of prime ideals.

   (b) Suppose that R is a domain such that every nonzero prime ideal in R contains a prime element. Show that R is a UFD. Hint: Use part (a). (Remark: the converse is also true, but not part of this problem.)

3. (a) Show that every A in the group $GL_N(\mathbb{C})$ that is of finite order is conjugate to a diagonal matrix.

   (b) If F is an algebraically closed field and $A \in GL_N(F)$ is of finite order, is A always conjugate to a diagonal matrix? Why or why not?

4. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 4$ and let K be a splitting field for $f(x)$ over $\mathbb{Q}$. Suppose that the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is S_n. If $\alpha \in K$ is a root of $f(x)$ show that $\alpha^n \notin \mathbb{Q}$.

5. Let k be a field and $k[x, y]$ the polynomial ring in two variables. Let I be the principle ideal generated by $x^2 - y^2(1 + y)$.

   (a) Show that $R = k[x, y]/I$ is an integral domain.

   (b) Describe the integral closure A of R in its field of fractions F explicitly by giving one or more elements of F that generate A over R, and prove your answer.

6. Let A, B be two finitely generated $\mathbb{Z}$-modules. Prove that $\mathrm{Tor}_n^{\mathbb{Z}}(A, B) = 0$ if $n > 1$.

7. Let V be a vector space over a field k and let $v_1, \ldots, v_n$ be linearly independent vectors in V. Let $p \geq 2$, and $w \in \wedge^p(V)$. Prove that $v_1 \wedge \cdots \wedge v_n \wedge w = 0$ if and only if there exist $y_1, \ldots, y_n \in \wedge^{p-1}V$ such that $w = \sum_{i=1}^n v_i \wedge y_i$.

8. Let R be an integral domain, let m be a maximal ideal in R and set $k = R/m$. Let P and Q be finitely generated R-modules and $\phi : P \to Q$ a map of R-modules. Suppose that the induced map $P/mP \to Q/mQ$ is a surjection of k-vector spaces. Prove that there is an element $f \in R$, whose image in k is nonzero, such that the map $\phi_f : P_f \to Q_f$ is a surjection of R_f-modules. (The subscript f denotes localization at the multiplicative set $\{1, f, f^2, f^3, \ldots\}$.)