AFFIRMATIVE ACTION WITH OVERLAPPING RESERVES

TAYFUN SÖNMEZ AND M. BUMIN YENMEZ

ABSTRACT. Affirmative action policies provide a balance between meritocracy and equity in a wide variety of real-life resource allocation problems. We study choice rules where meritocracy is attained by prioritizing individuals based on merit, and equity is attained by reserving positions for target groups of disadvantaged individuals. Focusing on overlapping reserves, the case where an individual can belong to multiple target groups, we characterize choice rules that satisfy maximal compliance with reservations, elimination of justified envy, and non-wastefulness. When an individual accommodates only one of the reserved positions, the *horizontal envelope choice rule* is the only rule to satisfy these three axioms. When an individual accommodates each of the reserved positions she qualifies for, there are complementarities between individuals. Under this alternative convention, and assuming there are only two target groups, such as women and minorities, *paired-admissions choice rules* are the only ones to satisfy the three axioms. Building on these results, we provide improved mechanisms for implementing a variety of recent reforms, including the 2015 school choice reform in Chile and 2012 college admissions reform in Brazil.

Keywords: Market design, matching, affirmative action, deferred acceptance
JEL codes: C78, D47, D63

Date: First version: December 2019.

Both Sönmez and Yenmez are affiliated with the Department of Economics, Boston College, 140 Commonwealth Ave, Chestnut Hill, MA, 02467. Emails: sonmezt@bc.edu, bumin.yenmez@bc.edu. Sönmez acknowledges the research support of Goldman Sachs Gives via Dalinc Ariburnu—Goldman Sachs Faculty Research Fund.

1. Introduction

While affirmative action remains to be a highly contested topic in the U.S., it has gained widespread acceptance in much of the world as public awareness of inequalities faced by various disadvantaged groups increased. As a result, affirmative action policies have been adopted worldwide in a variety of resource allocation problems, such as assignment of public school seats, government jobs, and legislative positions. Perhaps the most widely used class of affirmative action policies relies on reserving a certain fraction of seats for members of target beneficiary groups. In some applications, all items are identical, such as the allocation of legislative seats or decentralized admissions to a single university. In these applications, the outcome is determined through an (often rigorously defined) *choice rule*, which in general depends on the number of reserved seats for each target group, as well as other criteria such as a merit ranking of candidates (in applications such as allocation of school seats or assignment of government jobs) or the number of votes received by candidates (in applications such as allocation of legislative seats). These choice rules essentially capture policies which define the "property rights" over a set of homogenous indivisible goods. Hence, design of choice rules can be viewed as engineering of policies that govern allocation of these resources. In this paper, our focus is both the design and analysis of choice rules, as well as some of the direct policy implications of our results on three large-scale real-life applications from Chile, India, and Brazil.

While we extend our analysis to the more general case with heterogenous positions in Section 5, the starting point of our analysis is the more basic case with homogenous positions; i.e., the case where all the positions are identical. We consider affirmative action policies where the two main ingredients are, (1) an exogenous priority ranking of individuals, which would have solely dictated the choice of individuals in the absence of affirmative action, and (2) a number of reserved positions for each target group of individuals. Importantly, and deviating from much of the prior literature with a few exceptions, we allow each individual to be a member of multiple reserve-eligible target groups. That is, we consider the case of *overlapping reserves*, which has become increasingly widespread in real-life applications. For example, in Jordan and Pakistan there are reserved positions both for women and also for minorities for the seat allocation at their national assemblies (Htun, 2004).[1] All three applications presented in Section 6 also have this feature. In these applications, an individual who belongs to two target beneficiary groups, for example a minority woman, can benefit both from the reserved positions for women and reserved positions for minorities.

[1] In Jordanian House of Representatives, of the 110 seats 6 are reserved for women and 12 are reserved for Christians and Chechens/Circassians. In Pakistani National Assembly, of the 342 seats 60 are reserved for women and 10 are reserved for minorities (Htun, 2004).

Our approach is axiomatic, and throughout the paper we focus on choice rules that satisfy three basic properties, each of which is highly plausible in this framework:

(1) *Maximal compliance with reservations*: As many of the reserved positions as possible are to be allocated to the candidates from target groups.
(2) *Elimination of justified envy*: A lower-priority candidate is not to be assigned a position at the expense of a higher-priority candidate, unless doing so enables channelling strictly higher number of the reserved positions to qualified candidates.
(3) *Non-wastefulness*: All positions are to be allocated, to the extent there are sufficiently many candidates.

One distinction between various real-life implementations of overlapping reserves becomes critical for our modeling and analysis. Consider an individual who is a member of multiple reserve-eligible beneficiary groups. A key policy consideration is, whether an admission granted to this individual should count towards accommodating each one of the reserved positions she qualifies for, or only one of them. For example, suppose there is one position reserved for female candidates and one position reserved for minority candidates. When a minority woman is admitted, should her admission count towards accommodating both the women reserve as well as the minority reserve, or only one of them? Essentially, the distinction here is about the accounting convention adopted for enforcing reserves, and both conventions are observed in real-life applications. We refer to these two conventions as *one-to-all reserve matching* and *one-to-one reserve matching*, respectively.

While the choice of convention in enforcing reserves often depends on the specific application and it can be viewed as a policy decision, one of our main contributions is showing that it has important implications on the design. Even though either accounting convention brings its own challenges into design, adoption of the one-to-one reserve matching convention leads us to a much "cleaner" solution: Under this convention, the *horizontal envelope choice rule* introduced in Section 3 is the unique choice rule that satisfies non-wastefulness, maximal compliance with reservations, and elimination of justified envy (Theorem 1). Before elaborating on the intuition and details of this choice rule, it will be helpful to outline the challenges either convention brings into design.

Consider the one-to-all reserve matching convention, under which an individual upon admission counts towards each of the reserves she qualifies for. For example, admission of a minority woman counts towards accommodating both the women reserves and the minority reserves. The resulting challenge is well-known in the literature: Under this convention there are *complementarities* between various groups of individuals, which in general means admission of one of the individuals may hinge upon admission of another individual (see, e.g., Abdulkadiroğlu (2005)). For example, there are complementarities between majority men and minority women when there are reserved seats both for women and minorities, and thus the admission of a majority man may hinge upon the admission of a minority woman.[2] The presence of complementarities is one of the main challenges in market design, and it virtually rules out the design of a well-behaved choice rule.[3] Despite this challenge, in Section 4, we provide a full characterization of choice rules that satisfy our three axioms when the complementarities are in its most basic form when there are only two target groups. Analysis of this special case is still of interest, because in many practical applications there are only two reserve-eligible groups (such as women and minorities). In other applications, there may be complementarities between only two groups of individuals, even if there are more than two reserve-eligible groups. Our analysis in Section 6.3 of the 2012 *Law of Social Quotas* from Brazil is such an application.

Next consider the one-to-one reserve matching convention, under which an individual upon admission counts towards only one of her eligible reserves. The resulting challenge here is technical, as it introduces an additional dimension to the allocation problem. Under this convention, a solution is to determine not only who receives the reserved seats, but also which type of a reserved seat an individual is to receive if she belongs to multiple target groups. In other words, under this convention of enforcing reserves, there is a matching problem of individuals to different types of reserves, within the matching problem of individuals to positions. Fortunately, there is a clear solution to this more elaborate matching problem. We next provide an intuition for this solution, our proposed horizontal envelope choice rule, that naturally emerges in this version of the model.

Consider a small gathering where lunch is served. One of the two guests, Violet, is a vegetarian, whereas the second guest, Charlie, is flexible in his diet. Suppose there is one vegetable sandwich and one chicken sandwich available for the two guests. If Charlie is served his lunch prior to Violet, it would be a blunder to offer him the only vegetable sandwich, since that would have meant Violet has to skip her lunch. Charlie is flexible in his diet, whereas Violet is not, and serving the only vegetable sandwich to Charlie results in giving up this valuable flexibility. A more careful server would have planned ahead taking into Violet's dietary restriction into consideration, and thus would have offered Charlie the chicken sandwich utilizing the flexibility in his diet. Now consider allocation of public school seats to students. Suppose there is a female candidate Frida, a disabled female candidate Diana, one reserved seat for female candidates, and a reserved seat for disabled candidates. Just as it is not plausible to offer the vegetable sandwich to Charlie in the above scenario, it is not plausible to assign Diana the reserved seat for female candidates and consequently deny Frida a seat since she is not qualified for the reserved seat for disabled candidates. Both candidates can receive a reserved seat if Diana is instead assigned the

[2]Interestingly, Hughes (2011) provides empirical evidence from some countries where majority male candidates are supportive of minority female candidates.

[3]Notable exceptions are Ostrovsky (2008) and Alva (2019).

reserved seat for disabled candidates. Ironically, many real-life applications do not utilize the flexibility generated by candidates who qualify for multiple types of reserves, and thus suffer from the very shortcoming we illustrate in these two scenarios. For example, as we present in Section 6.1, the brand new school choice system in Chile suffers from this exact shortcoming, precisely for the reason we illustrate here. A better design would not give up the flexibility generated by candidates who qualify for multiple reserves, and instead capitalize on it. This is the basic idea under our proposed horizontal envelope choice rule. We argue that this choice rule is the only plausible choice rule under the one-to-one reserve matching convention, for it is the unique choice rule that satisfies fairly basic desiderata.

When the allocation problem involves heterogenous goods, the design is more involved, because the system has to determine not only the set of recipients who receive an item, but also which recipient receives which item. Over the last couple of decades, the celebrated *agent-proposing deferred acceptance algorithm* (Gale and Shapley, 1962) has gained acceptance worldwide as the mechanism of choice for this purpose in different applications, including some with diversity constraints.[4] Hence, it is natural to rely on the same approach in this framework as well. Analysis of the more general version of the problem with heterogeneous goods reveals another advantage of the accounting convention of one-to-one reserve matching, compared to one-to-all reserve matching. The *agent-proposing deferred acceptance algorithm* is well-behaved only if the choice rule used by each institution satisfies a technical condition known as the *substitutes* condition. While the horizontal envelope choice rule satisfies substitutability (Proposition 4), there is no choice rule that satisfies our three properties along with substitutability under the one-to-all reserve matching convention (Proposition 5).

The following summary of our main theoretical results is helpful to compare and contrast the two alternative conventions for enforcing reserves.

(1) Under the one-to-one reserve matching convention:
   (a) There is a unique choice rule, namely the horizontal envelope choice rule, that satisfies maximal compliance with reservations, elimination of justified envy, and non-wastefulness.
   (b) The horizontal envelope choice rule satisfies the substitutes condition, and it can be jointly implemented with the agent-proposing deferred acceptance algorithm when items are heterogeneous.

[4]This approach was initially proposed for school choice mechanisms in Balinski and Sönmez (1999), Abdulkadiroğlu and Sönmez (2003), and it was first put into practice with the subsequent school choice reforms in New York City (Abdulkadiroğlu et al., 2005b) and Boston (Abdulkadiroğlu et al., 2005a). Other models where versions of the agent-proposing deferred acceptance algorithm is proposed includes, large matching markets (Kojima and Pathak, 2009; Azevedo and Leshno, 2016; Che et al., 2019) refugee resettlement (Delacrétaz et al., 2016), and airline landing slot assignment (Schummer and Abizada, 2017).

(2) Under the one-to-all reserve matching convention, and assuming there are only two target beneficiary groups:
   (a) A choice rule satisfies maximal compliance with reservations, elimination of justified envy, and non-wastefulness if, and only if, it is a selection from the *paired-admissions choice correspondence*.
   (b) Presence of complementarities is reflected in the structure of selections from the paired-admissions choice correspondence: This class of choice rules has an optimal selection for individuals who are either members of both target groups or a member of neither target group. This selection is also pessimal for individuals who are members of only one of the target groups. Similarly, there is an optimal selection for individuals who are members of only one of the target groups, which is also the pessimal selection for individuals who are either members of both target groups or a member of neither target group.
   (c) There is no selection from the paired-admissions choice correspondence that satisfies the substitutes condition. Therefore, of the choice rules that satisfy non-wastefulness, maximal compliance with reservations, and elimination of justified envy, none of them can be jointly implemented with the agent-proposing deferred acceptance algorithm when items are heterogeneous.

Our results have direct policy implications for several real-life applications of affirmative action. Some of these policy implications, presented in Section 6, include:

(1) Bringing into light two shortcomings of the recently designed school choice mechanism for Chile, as well as the design of an improved mechanism that complies with the 2015 School Inclusion Law,
(2) bringing into light two shortcomings of a Supreme Court-mandated affirmative action procedure from India, as well as its unique improvement that satisfies our three axioms, and
(3) the provision of a class of college admissions mechanisms for Brazil that complies with *The Law of Social Quotas*, without imposing any additional restrictions beyond those mandated by the law.

1.1. Related Literature. There is a large literature on market design under various classes of distributional constraints such as minimum guarantee reserves (or lower quotas), upper quotas, and regional quotas.[5] What differentiates our paper from the others is the following four features:

(1) Our focus is the analysis and design of affirmative action policies under overlapping reserves, a version of the problem that is neither well-understood nor analyzed.
(2) In previous studies, which adopted the one-to-one reserve matching convention, individuals are either assumed to have strict preferences between reserved seats of different types (as in Aygün and Turhan (2016) and Kurata et al. (2017)) or their indifferences are broken through fixed tie-breaking rules (as in Baswana et al. (2018) and Correa et al. (2019)). To the best of our knowledge, our proposed horizontal envelope choice rule is the first choice rule that utilizes the flexibility in reserve matching for an improved design.
(3) Relying on three intuitive axioms, we provide full axiomatic characterizations of choice rules for both reserve matching conventions considered in our paper. To the best of our knowledge, these are the first axiomatic characterizations under overlapping reserves.[6]
(4) Our results have direct policy implications for large scale applications from Chile, India, and Brazil.

The one paper that is closest to our study in its objective is Kurata et al. (2017). As in our paper, Kurata et al. (2017) also consider affirmative action with overlapping reserves, and due to possible non-existence of *stable* matchings under one-to-all reserve matching convention,[7] they propose the one-to-one matching convention. This is where our approach diverges with Kurata et al. (2017). In order to directly invoke the analysis of Hatfield and Milgrom (2005) on the *matching with contracts* model, they assume that individuals have strict preferences on reserved seats of different types. This assumption together with the one-to-one reserve matching convention assures that their model becomes a special case of the matching with contracts model, and all positive results of Hatfield and Milgrom (2005) follow directly. There is one footnote in Kurata et al. (2017) which states

> If a student is indifferent between different seats of the same school, we can break the ties arbitrarily to form a strict preference and apply our mechanism, which is still

[5]A very incomplete list includes Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu (2005), Biro et al. (2010), Kojima (2012), Budish et al. (2013), Hafalir et al. (2013), Westkamp (2013), Ehlers et al. (2014), Echenique and Yenmez (2015), Kamada and Kojima (2015, 2017, 2018), Aygün and Bó (2016), Aygün and Turhan (2016), Bó (2016), Delacrétaz et al. (2016), Doğan (2016), Dur et al. (2016, 2018), Kominers and Sönmez (2016), Fragiadakis and Troyan (2017), Goto et al. (2017), Kurata et al. (2017), Hafalir et al. (2018), Kojima et al. (2018), and Sönmez and Yenmez (2019a,b).

[6]Echenique and Yenmez (2015) provide characterizations of choice rules when each individual is eligible for only one reserve. Other choice rules used in matching are characterized in Chambers and Yenmez (2018) and Doğan et al. (2019).

[7]Their notion of stability is equivalent to elimination of justified envy together with non-wastefulness under the one-to-all reserve matching convention.

strategy-proof and obtains a stable matching. However, we can no longer guarantee student-optimality.

Indeed their suggestion of breaking ties arbitrarily to form a strict preference, and applying the resulting choice rule in conjunction with the agent-proposing deferred acceptance algorithm was followed by the research group that have recently designed the Chilean school choice mechanism. What is misleading in their footnote is that, their stability notion fails to allow for schools to utilize the flexibility to move students between different types of reserves to accommodate either higher priority students or to fill a greater number of reserved seats. Therefore, while their stability notion is equivalent to the combination of elimination of justified envy and non-wastefulness for the one-to-all reserve matching convention, it is a weaker notion under the one-to-one reserve matching convention. In particular, when students are indifferent between different seats of a given school, their proposed choice rule fails not only maximal compliance with reservations (see Example 4 in Section 6.1), but also elimination of justified envy (see Example 3 in Section 6.1).

2. Model: Homogeneous Jobs

There exist a set of individuals $\mathcal{I}$ and a set of traits $\mathcal{T}$. Each individual $i \in \mathcal{I}$ has a set of traits $\tau(i) \subseteq \mathcal{T}$. A trait can specify gender, race, or socioeconomic status of an individual. There are q identical positions to allocate, which may correspond to positions at a job, seats at a school, or seats for legislative organs. Individuals are strictly ranked according to a priority order π. Therefore, $i \; \pi \; j$ means that individual $i \in \mathcal{I}$ has a strictly higher priority than individual $j \in \mathcal{I}$. Without loss of generality, we assume that all individuals are eligible for this job.

For every trait t, $r_t \in \mathbb{N}$ number of positions are reserved for individuals with trait t. We assume that the number of reserved positions is less than the number of all positions, i.e., $\sum_{t \in \mathcal{T}} r_t \leq q$. For any set of individuals $I \subseteq \mathcal{I}$ and trait $t \in \mathcal{T}$, let I_t denote the set of individuals in I who have trait t, i.e., $I_t = \{i \in I | t \in \tau(i)\}$.

A **reservation market** is a tuple $\langle \mathcal{I}, \mathcal{T}, \tau, \pi, q, (r_t)_{t \in \mathcal{T}} \rangle$.

A **choice rule** C is a function that takes a set of individuals $I \subseteq \mathcal{I}$ and produces a subset $C(I) \subseteq I$ for a given reservation market. A choice rule may depend on the number of reserved positions for each trait and also the number of positions. For notational brevity, we omit this dependence.

We consider the following two conventions for implementing reserves:

(1) **One-to-one reserve matching:** Upon admission, an individual with multiple traits accommodates only one of the reserves she qualifies for.
(2) **One-to-all reserve matching:** Upon admission, an individual with multiple traits accommodates each of the reserves she qualifies for.

Both conventions are used in real-life applications.

3. One-to-One Reserve Matching & Horizontal Envelope Choice Rule

In this section, we adopt the one-to-one reserve matching convention. That is, throughout this section an individual is assumed to accommodate reserves for only one of her traits.

Fix a reservation market $\langle \mathcal{I}, \mathcal{T}, \tau, \pi, q, (r_t)_{t\in\mathcal{T}} \rangle$, and consider a set of applicants $I \subseteq \mathcal{I}$. Construct the following two-sided **reservation graph**. On one side of the graph, there are individuals in I. On the other side, there are reserved positions, i.e., there is a position for each reservation. Therefore, for each trait t, there are r_t positions and, there are $\sum_{t\in\mathcal{T}} r_t$ positions in total. An individual and a position are connected if the individual has the trait of the position and thus they can be matched with each other. A one-to-one matching of individuals with positions **has maximum cardinality in reserve matching** if there exists no other one-to-one matching that assigns strictly higher number of reserved positions to individuals. Let $n(I)$ denote this maximum number of reserved positions that can be assigned to individuals. Say that an individual i **increases reserve utilization of I**, if $n(I \cup \{i\}) = n(I) + 1$.

Definition 1. *A set of individuals $I' \subseteq I$ **maximally complies with reservations for I**, if, there exists a one-to-one matching of individuals in I' to the reserved positions with maximum cardinality $n(I)$ in reserve matching.*

*A choice rule C **maximally complies with reservations** if, for every set $I \subseteq \mathcal{I}$, $C(I)$ maximally complies with reservations for I.*

Consider the following **horizontal envelope choice rule**:

Choice Rule $\mathbf{C}^{\boxtimes}$
Step 1.1: Choose the highest priority individual with a trait that has a reserved position. Let i_1 be this individual and I_1 be the set including only this individual. If there exists no such individual, go to Step 2.
Step 1.k ($\mathbf{k} \in [\mathbf{2}, \sum_{\mathbf{t}\in\mathcal{T}} \mathbf{r_t}]$): Starting from the individual who has the next highest priority after i_{k-1}, check one by one if the next individual increases reserve utilization of I_{k-1}.[8] If she does, choose this individual and denote her by i_k. In this case, let $I_k = I_{k-1} \cup \{i_k\}$ be the set of individuals chosen so far. Otherwise, if no such individual exists, go to Step 2.
Step 2: For unfilled positions, choose remaining individuals with the highest priority until all positions are filled or there are no unchosen individuals remaining.

[8]This can be done with various computationally efficient algorithms, see, for example, the bipartite cardinality matching algorithm (Lawler, 2001, Page 195).

When the number of individuals is less than q, this procedure selects all individuals. Otherwise, if there are more than q individuals, then it chooses a set with q individuals.

We illustrate the horizontal envelope choice rule with the following example.

Example 1. Consider the following reservation market:

- $\mathcal{I} = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$,
- $\mathcal{T} = \{t_1, t_2, t_3\}$,
- $i_1 \ \pi \ i_2 \ \pi \ i_3 \ \pi \ i_4 \ \pi \ i_5 \ \pi \ i_6 \ \pi \ i_7$,
- $\tau(i_1) = \emptyset$, $\tau(i_2) = \{t_1, t_2, t_3\}$, $\tau(i_3) = \emptyset$, $\tau(i_4) = \{t_1, t_2\}$, $\tau(i_5) = \{t_1\}$, $\tau(i_6) = \{t_3\}$, $\tau(i_7) = \{t_2\}$,
- $r_{t_1} = r_{t_2} = r_{t_3} = 1$, and $q = 5$.

The reservation graph for this market has one position for each trait and three positions in total. An individual is connected with a position if the individual has the corresponding trait. The reservation graph is depicted in Figure 1.

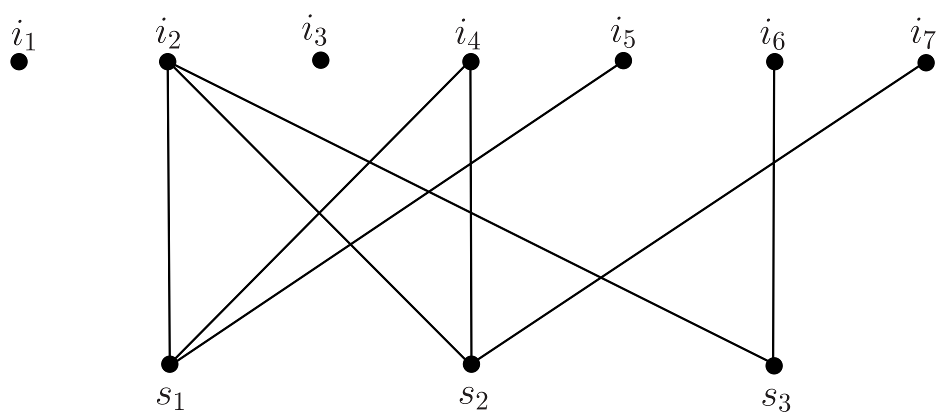

FIGURE 1. The reservation graph of the market in Example 1. Vertex s_k represents the reserved position for trait t_k for $k \in \{1, 2, 3\}$.

Let $\mathcal{I}$ be the set of applicants. The horizontal envelope choice rule works as follows. Only individuals i_2, i_4, i_5, i_6, and i_7 are qualified to receive a position at the first step because they have at least one trait. At Step 1.1, individual i_2 is selected because she is the highest priority individual who qualifies for a reserved position. At Step 1.2, individual i_4 is selected because she is the highest priority individual who can simultaneously be matched to a reserved position along with i_2. For example, i_2 can be matched with s_1 and i_4 can be matched with s_2 (see the dashed matching in Figure 2).

At Step 1.3, individual i_5 is selected because she is the highest ranked individual who can be matched to a reserved position together with i_2 and i_4. However, the implementation of such a matching requires that i_2 and i_4 are matched with different positions than the

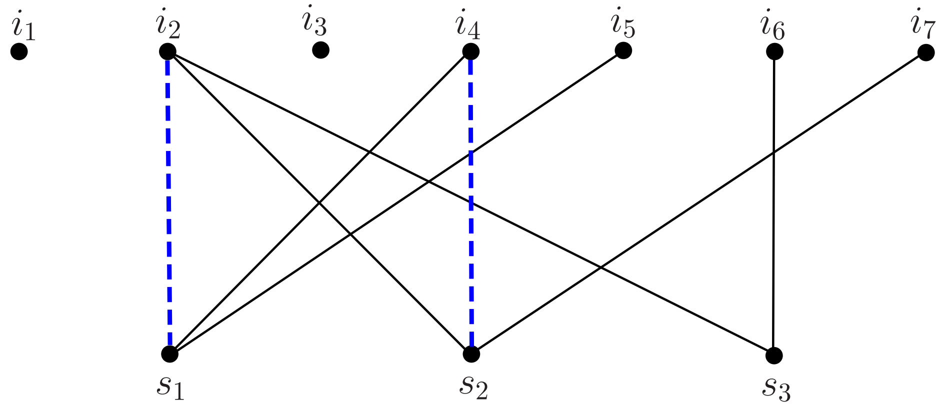

FIGURE 2. Individuals i_2 and i_4 can be matched with different positions in the reservation graph.

dashed matching shown in Figure 2. To be more precise, i_2 can be matched with s_3, i_4 can be matched with s_2, and i_5 can be matched with s_1 (see the dotted matching in Figure 3).

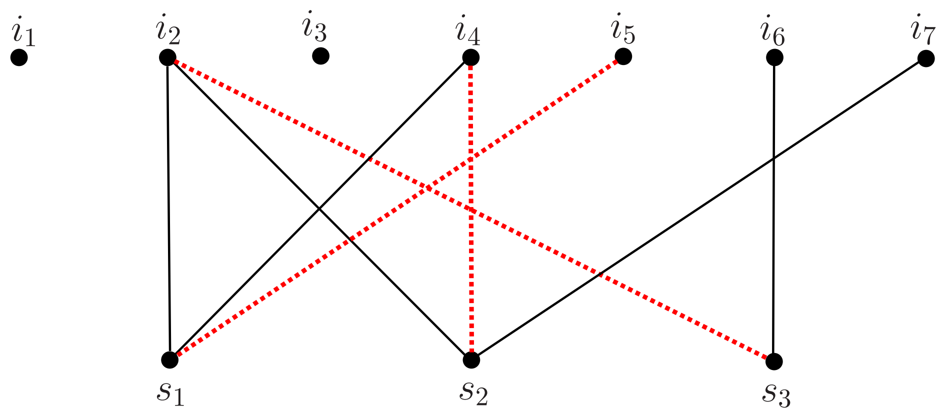

FIGURE 3. Individuals i_2, i_4, and i_5 can be matched with different positions in the reservation graph.

No remaining individuals can be matched with a reserved position together with i_2, i_4, and i_5, so we go to Step 2. At Step 2, individuals i_1 and i_3 are chosen because there are two vacant positions and they have the highest priority among the remaining individuals. Therefore, $C^{\boxtimes}(I) = \{i_1, i_2, i_3, i_4, i_5\}$. □

Our first result shows that the horizontal choice rule $C^{\boxtimes}$ selects higher priority individuals than any other choice rule that maximally complies with reservations.

Proposition 1. *Let C be a choice rule that maximally complies with reservations. Then, for every set of individuals I,*

(1) $|C(I)| \leq |C^{\boxtimes}(I)|$ and

(2) *for every* $k \leq |C(I)|$, *if* i *is the individual with the k-th highest priority in* $C^{\boxtimes}(I)$ *and* i' *is the individual with the k-th highest priority in* $C(I)$, *then*

$$i \; \pi \; i'.$$

Next we extend a standard fairness axiom to the present model.

Definition 2. *A choice rule* C ***eliminates justified envy*** *if for every* $I \subseteq \mathcal{I}$, $i \in C(I)$, *and* $i' \in I \setminus C(I)$,

$$i' \; \pi \; i \implies n\left((C(I) \setminus \{i\}) \cup \{i'\}\right) < n(C(I)).$$

In words, if a high-priority individual i' is rejected even though a low-priority individual i is chosen, then it must be the case that replacing i with i' in the chosen set decreases the number of reserved positions that can be filled. When this condition is violated, we say that there is justified envy, which means that there exist a set of individuals I and two individuals $i, i' \in I$ such that

(1) $i \in C(I)$,
(2) $i' \notin C(I)$,
(3) $n\left((C(I) \setminus \{i\}) \cup \{i'\}\right) \geq n(C(I))$, and
(4) $i' \; \pi \; i$.

Therefore, when there is justified envy, a low-priority individual i can be replaced with a high-priority individual i' without decreasing the number of reserved positions that can be filled.

Our next axiom, standard in the analysis of choice rules, is a weak efficiency property.

Definition 3. *A choice rule* C *is* ***non-wasteful*** *if for every* $I \subseteq \mathcal{I}$,

$$|C(I)| = \min\{|I|, q\}.$$

Equivalently, non-wastefulness requires that an individual is rejected only when all positions are allocated.

We next present one of our main results, a characterization of the horizontal envelope choice rule.

Theorem 1. *Consider the one-to-one reserve matching convention. A choice rule maximally complies with reservations, eliminates justified envy, and is non-wasteful if, and only if, it is the horizontal envelope choice rule.*

4. One-to-All Reserve Matching with Two Traits

In this section, we adopt the one-to-all reserve matching convention. That is, throughout this section an individual is assumed to accommodate each of the reserves for her traits.

Given the complementarities introduced under this convention, analysis of the model in its full generality is not tractable. Hence, we consider the simplest version of this model with only two traits. This simplified version of the model is still of interest, because it is fairly common in real-life applications. Let $\mathcal{T} = \{t_1, t_2\}$ be the set of traits. Suppose that $r_{t_1} \leq q$ and $r_{t_2} \leq q$.

Parallel to the analysis in Section 3, we next characterize the class of choice rules that satisfy our three axioms under this alternative reserve matching convention for the case of two traits. However, even for this simplified case, the description of the resulting *paired-admissions choice rules* is fairly involved. While the description and analysis of this class is based on a "brute-force" case-by-case analysis, thus lacking the elegance of the analysis horizontal envelope choice rule presented in Section 3, it provides us with an opportunity to compare and contrast the two reserve matching conventions considered in our paper. This comparison may be valuable for a real-life design, if there is flexibility to choose between the two conventions. Moreover, despite the "brute-force" description and analysis of paired-admissions choice rules, their structure is still very intuitive and it heavily utilizes the complementarities between groups of individuals.

We have to modify a few definitions under the one-to-all reserve matching convention. Since an individual who has both traits counts against reserved positions for both traits, an outcome under this convention is a many-to-one matching of individuals to reserved (or open) positions, where an individual who has both traits can be matched with a reserved position for each trait. A many-to-one matching of at most q individuals with positions **has maximum cardinality in reserve matching** if there exists no other many-to-one matching that assigns at most q individuals to strictly higher number of reserved positions. Let $n(I)$ denote this maximum number of reserved positions that can be filled.

With this adjustment, maximal compliance with reservations can be defined as before:

Definition 4. *A set of individuals $I' \subseteq I$ **maximally complies with reservations for** $\mathbf{I}$, if, there exists a many-to-one matching of individuals in I' to the reserved positions with maximum cardinality $n(I)$ in reserve matching.*

*A choice rule C **maximally complies with reservations** if, for every set $I \subseteq \mathcal{I}$, $C(I)$ maximally complies with reservations for I.*

We utilize the following additional notation to introduce the class of paired-admissions choice rules, given through a sequential procedure. Let

- $r_t(k)$ denote the number of trait-t reserved positions that are not filled, and
- $q(k)$ denote the number of positions that are not filled

prior to Step k of the procedure. Furthermore, let

$$\Delta(k) = r_{t_1}(k) + r_{t_2}(k) - q(k)$$

denote the number of unfilled reserved positions in excess of the unfilled capacity. At Step 1, $q(1) = q$ and, $r_t(1) = r_t$ for every $t \in \mathcal{T}$.

For a given set of individuals I who are already selected and $i \notin I$, let $\delta(i|I)$ denote the contribution of individual i to accommodate the unfilled reserved positions in excess of the unfilled capacity. Formally,

$$\delta(i|I) = \begin{cases} 1, & \text{if } n(I \cup \{i\}) = n(I) + 2 \\ 0, & \text{if } n(I \cup \{i\}) = n(I) + 1 \\ -1, & \text{otherwise.} \end{cases}$$

Therefore, when I is the set of individuals chosen before Step k and i is chosen at Step k, we have

$$\Delta(k+1) = \Delta(k) - \delta(i|I).$$

Let I be the set of applicants and $I(k)$ denote the set of individuals not chosen prior to Step k. Set $I(1) = I$. Therefore, $I \setminus I(k)$ is the set of individuals chosen prior to Step k. The following choice procedure has two phases. At its (main) *individual-admissions phase*, one individual is chosen at each step. Once (and if) the procedure enters its *paired-admissions phase*, on the other hand,

(1) it will never go back to the individual-admissions phase,
(2) complementarities become important,
(3) admissions are carried out in pairs, and
(4) multiple sets of individuals are identified for the remaining positions.

Each set identified under the paired-admissions phase indicates a group of individuals that can be collectively chosen. Therefore, if the procedure goes on to the paired-admissions phase, it identifies a choice correspondence that has multiple sets of individuals as the outcome. In contrast, if the procedure terminates at its individual admissions round, it produces a unique set of individuals as its outcome. Figure 4 shows how each case in this procedure is determined.

Paired-Admissions Choice Correspondence

At each Step k, consider the set of individuals who are not yet chosen, and choose one or more of them depending on the case below. Terminate the procedure when no individuals or positions remain.

Under all cases with the exception of the last case, only one individual is selected. Under the last case, multiple sets of individuals are identified, each as a possible group to select together. It is due to this last case that the procedure is a correspondence rather than a function.

Individual-Admissions Phase

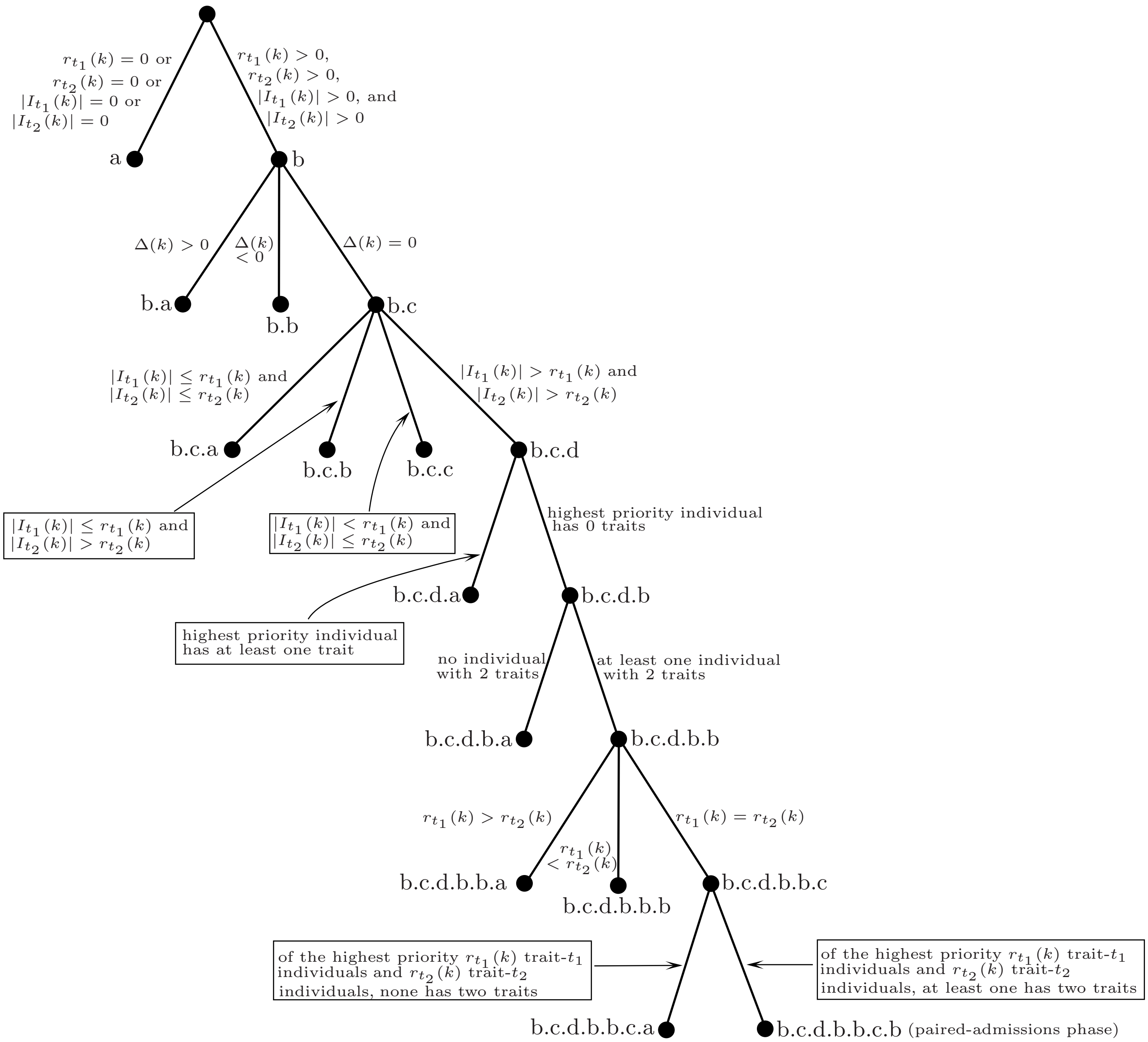

FIGURE 4. Paired-admissions choice correspondence cases represented as a tree. Each terminal node corresponds to a case at which one or more individuals are chosen.

Case a

(1a) $r_{t_1}(k) = 0$ or $r_{t_2}(k) = 0$ or $|I_{t_1}(k)| = 0$ or $|I_{t_2}(k)| = 0$

Choose the highest priority individual i, among those with the highest value of $\delta(i|I \setminus I(k))$.

Case b.a

(1b) $r_{t_1}(k) > 0$, $r_{t_2}(k) > 0$, $|I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$

(2a) $\Delta(k) > 0$

Choose the highest priority individual i, among those with the highest value of $\delta(i|I \setminus I(k))$.

Case b.b

(1b) $r_{t_1}(k) > 0, r_{t_2}(k) > 0, |I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$

(2b) $\Delta(k) < 0$

Choose the highest priority individual.

Case b.c.a

(1b) $r_{t_1}(k) > 0, r_{t_2}(k) > 0, |I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$

(2c) $\Delta(k) = 0$

(3a) $|I_{t_1}(k)| \leq r_{t_1}(k)$ and $|I_{t_2}(k)| \leq r_{t_2}(k)$

Choose the highest priority individual who has at least one trait.

Case b.c.b

(1b) $r_{t_1}(k) > 0, r_{t_2}(k) > 0, |I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$

(2c) $\Delta(k) = 0$

(3b) $|I_{t_1}(k)| \leq r_{t_1}(k)$ and $|I_{t_2}(k)| > r_{t_2}(k)$

Choose the highest priority individual who has trait t_1.

Case b.c.c

(1b) $r_{t_1}(k) > 0, r_{t_2}(k) > 0, |I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$

(2c) $\Delta(k) = 0$

(3c) $|I_{t_1}(k)| > r_{t_1}(k)$ and $|I_{t_2}(k)| \leq r_{t_2}(k)$

Choose the highest priority individual who has trait t_2.

Case b.c.d.a

(1b) $r_{t_1}(k) > 0, r_{t_2}(k) > 0, |I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$

(2c) $\Delta(k) = 0$

(3d) $|I_{t_1}(k)| > r_{t_1}(k)$ and $|I_{t_2}(k)| > r_{t_2}(k)$

(4a) the highest priority individual has at least one trait

Choose the highest priority individual.

Case b.c.d.b.a

(1b) $r_{t_1}(k) > 0, r_{t_2}(k) > 0, |I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$

(2c) $\Delta(k) = 0$

(3d) $|I_{t_1}(k)| > r_{t_1}(k)$ and $|I_{t_2}(k)| > r_{t_2}(k)$

(4b) the highest priority individual has no traits

(5a) there exists no individual with both traits

Choose the highest priority individual i, among those with the highest value of $\delta(i|I \setminus I(k))$.

Case b.c.d.b.b.a

(1b) $r_{t_1}(k) > 0$, $r_{t_2}(k) > 0$, $|I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$
(2c) $\Delta(k) = 0$
(3d) $|I_{t_1}(k)| > r_{t_1}(k)$ and $|I_{t_2}(k)| > r_{t_2}(k)$
(4b) the highest priority individual has no traits
(5b) there exists an individual with both traits
(6a) $r_{t_1}(k) > r_{t_2}(k)$

Choose the highest priority individual who has trait t_1.

Case b.c.d.b.b.b

(1b) $r_{t_1}(k) > 0$, $r_{t_2}(k) > 0$, $|I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$
(2c) $\Delta(k) = 0$
(3d) $|I_{t_1}(k)| > r_{t_1}(k)$ and $|I_{t_2}(k)| > r_{t_2}(k)$
(4b) the highest priority individual has no traits
(5b) there exists an individual with both traits
(6b) $r_{t_1}(k) < r_{t_2}(k)$

Choose the highest priority individual who has trait t_2.

Case b.c.d.b.b.c.a

(1b) $r_{t_1}(k) > 0$, $r_{t_2}(k) > 0$, $|I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$
(2c) $\Delta(k) = 0$
(3d) $|I_{t_1}(k)| > r_{t_1}(k)$ and $|I_{t_2}(k)| > r_{t_2}(k)$
(4b) the highest priority individual has no traits
(5b) there exists an individual with both traits
(6c) $r_{t_1}(k) = r_{t_2}(k)$
(7a) Of the remaining $r_{t_1}(k)$ highest priority individuals with trait t_1 along with the remaining $r_{t_2}(k)$ highest priority individuals with trait t_2, at least one of them has both traits.

Choose the highest priority individual who has both traits.

Paired-Admissions Phase

Case b.c.d.b.b.c.b

(1b) $r_{t_1}(k) > 0$, $r_{t_2}(k) > 0$, $|I_{t_1}(k)| > 0$, and $|I_{t_2}(k)| > 0$
(2c) $\Delta(k) = 0$
(3d) $|I_{t_1}(k)| > r_{t_1}(k)$ and $|I_{t_2}(k)| > r_{t_2}(k)$
(4b) the highest priority individual has no traits
(5b) there exists an individual with both traits
(6c) $r_{t_1}(k) = r_{t_2}(k)$

(7b) Of the remaining $r_{t_1}(k)$ highest priority individuals with trait t_1 along with the remaining $r_{t_2}(k)$ highest priority individuals with trait t_2, none of them has both traits.

If the procedure reaches this phase, it terminates as follows: Let $r \equiv r_{t_1}(k) = r_{t_2}(k)$. Let $p_0 = \emptyset$, and for any $n \in \{1, \ldots, r\}$, let the pair of individuals $p_n = \{i_n, j_n\}$ be such that

(1) i_n is the n-th highest priority individual with trait t_1, and
(2) j_n is the n-th highest priority individual with trait t_2.

Let $m^* \leq r$ be the greatest positive integer such that

(1) there are at least m^* individuals with no traits,
(2) there are at least m^* individuals with both traits, and
(3) the m^*-th highest priority individual with no trait has higher priority than both individuals in p_{r-m^*+1}.

Since the highest priority individual has no traits and there exists at least one individual with both traits under this case, $m^* \geq 1$. Let $q_0 = \emptyset$, and for any $m \in \{1, \ldots, m^*\}$, let the pair of individuals $q_n = \{i'_n, j'_n\}$ be such that

(1) i'_n is the n-th highest priority remaining individual with no traits, and
(2) j'_n is the n-th highest priority individual with both traits.

For each $m \in \{0, 1, \ldots, m^*\}$, the set of pairs $q_1 \cup \ldots \cup q_m \cup p_1 \cup \ldots \cup p_{r-m}$ can be chosen in this last phase.[9]

If this procedure terminates at Case b.c.d.b.b.c.b (i.e., at the paired-admissions phase), then there are $m^* + 1$ different sets of individuals that can be chosen. However, if it terminates at any other case, then there is only one set of individuals that is chosen. We call any selection from the paired-admissions choice correspondence a **paired-admissions choice rule**. The following example illustrates the paired-admissions choice correspondence.

Example 2. Consider the following reservation market:

- $\mathcal{I} = \{i_1, i_2, \ldots, i_{16}\}$,
- $i_1 \ \pi \ i_2 \ \pi \ \ldots \ \pi \ i_{16}$,
- $\tau(i) = \emptyset$ for $i \in \{i_1, i_2, i_5, i_8, i_{15}\}$, $\tau(i) = \{t_1\}$ for $i \in \{i_4, i_6, i_{10}, i_{13}\}$, $\tau(i) = \{t_2\}$ for $i \in \{i_3, i_7, i_9, i_{11}\}$, $\tau(i) = \{t_1, t_2\}$ for $i \in \{i_{12}, i_{14}, i_{16}\}$,
- $r_{t_1} = 3$, $r_{t_2} = 4$, and $q = 8$.

Let $\mathcal{I}$ be the set of applicants. The paired-admissions choice correspondence works as follows. At Step 1, we are at Case b.b because

(1b) $r_{t_1}(1) = 3 > 0$, $r_{t_2}(1) = 4 > 0$, $|\mathcal{I}_{t_1}(1)| = 7 > 0$, $|\mathcal{I}_{t_2}(1)| = 7 > 0$, and

[9]More precisely, the resulting set of individuals can be selected for any set of pairs $q_1 \cup \ldots \cup q_m \cup p_1 \cup \ldots \cup p_{r-m}$.

(2b) $\Delta(1) = r_{t_1}(1) + r_{t_2}(1) - q(1) = 3 + 4 - 8 = -1 < 0$.

Therefore, the highest priority individual, i_1, is chosen at the first step.

At Step 2, we are at Case b.c.d.b.b.b because

(1b) $r_{t_1}(2) = 3 > 0$, $r_{t_2}(2) = 4 > 0$, $|\mathcal{I}_{t_1}(2)| = 7 > 0$, $|\mathcal{I}_{t_2}(2)| = 7 > 0$,
(2c) $\Delta(2) = r_{t_1}(2) + r_{t_2}(2) - q(2) = 3 + 4 - 7 = 0$,
(3d) $|\mathcal{I}_{t_1}(2)| = 7 > 3 = r_{t_1}(2)$, $|\mathcal{I}_{t_2}(2)| = 7 > 4 = r_{t_2}(2)$,
(4b) the highest priority individual, i_2, has no traits,
(5b) there exists an individual, e.g., i_{12}, with both traits, and
(6b) $r_{t_1}(2) = 3 < 4 = r_{t_2}(2)$.

Thus, the highest priority individual who has trait t_2, i_3, is chosen at the second step.

At Step 3, we are at Case b.c.d.b.b.c.b (the paired-admissions phase) because

(1b) $r_{t_1}(3) = 3 > 0$, $r_{t_2}(3) = 3 > 0$, $|\mathcal{I}_{t_1}(3)| = 7 > 0$, $|\mathcal{I}_{t_2}(3)| = 6 > 0$
(2c) $\Delta(3) = r_{t_1}(3) + r_{t_2}(3) - q(3) = 3 + 3 - 6 = 0$,
(3d) $|\mathcal{I}_{t_1}(3)| = 7 > 3 = r_{t_1}(3)$, $|\mathcal{I}_{t_2}(3)| = 6 > 3 = r_{t_2}(3)$,
(4b) the highest priority individual, i_2, has no traits,
(5b) there exists an individual, e.g., i_{12}, with both traits,
(6c) $r_{t_1}(3) = r_{t_2}(3)$, and
(7b) of the remaining $r_{t_1}(3) = 3$ highest priority individuals with trait t_1, $\{i_4, i_6, i_{10}\}$, and $r_{t_2}(3) = 3$ highest priority individuals with trait t_2, $\{i_7, i_9, i_{11}\}$, none of them has both traits.

Since $r = 3$, we form pairs $p_1 = \{i_4, i_7\}$, $p_2 = \{i_6, i_9\}$, and $p_3 = \{i_{10}, i_{11}\}$ that have an individual with only trait t_1 and an individual with only trait t_2. Furthermore, since the second highest priority individual with no traits, i_5, has a higher priority than both individuals in p_2 but the third highest priority individual with no traits, i_8, does not have a higher priority than both individuals in p_1, we have $m^* = 2$. Therefore, $q_1 = \{i_2, i_{12}\}$ and $q_2 = \{i_5, i_{14}\}$. At this step, three sets of individuals can be chosen:

- $p_1 \cup p_2 \cup p_3$,
- $q_1 \cup p_1 \cup p_2$, and
- $q_1 \cup q_2 \cup p_1$.

As a result, the paired-admissions choice correspondence chooses the following sets when $\mathcal{I}$ is the set of applicants:

- $\{i_1, i_3, i_4, i_6, i_7, i_9, i_{10}, i_{11}\}$,
- $\{i_1, i_2, i_3, i_4, i_6, i_7, i_9, i_{12}\}$, and
- $\{i_1, i_2, i_3, i_4, i_6, i_7\}$.

Figure 5 illustrates the outcome.

□

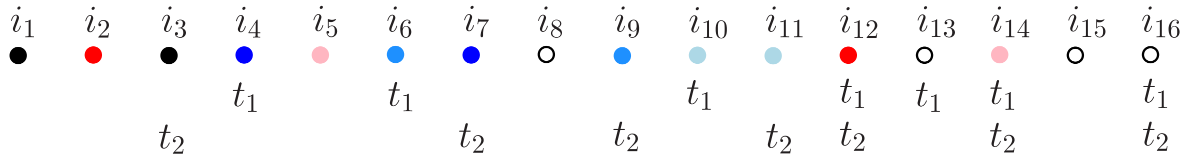

FIGURE 5. An illustration of the paired-admissions correspondence in Example 2. The individuals selected in the individual-admissions phase are denoted by solid nodes, the individuals selected in the paired-admissions phase are denoted by colored nodes such that individuals in the same pair have the same color, and the individuals who are never selected are represented by hollow nodes.

The main result of this section is the following counterpart to Theorem 1.

Theorem 2. *Consider the one-to-all reserve matching convention. A choice rule maximally complies with reservations, eliminates justified envy, and is non-wasteful if, and only if, it is a paired-admissions choice rule.*

The paired-admissions choice correspondence has two extremal choice rules, each of which favors individuals with certain traits. The first one, $C^{\min\max}$, favors individuals with no traits along with individuals with both traits. To be more specific, at the paired-admissions phase of this selection from the procedure, the maximum number of pairs that include an individual with no traits and an individual with both traits are chosen. Therefore, $q_1 \cup \ldots \cup q_{m^*} \cup p_1 \cup \ldots \cup p_{r-m^*}$ is the set of individuals chosen at the paired-admissions phase. Similarly, there is a paired-admissions choice rule, $C^{\max\min}$, that favors individuals with exactly one trait. At the paired-admissions phase of this alternative selection from the procedure, this choice rule selects all pairs that have an individual with trait t_1 along with an individual with trait t_2. Therefore, at the paired-admissions phase, the set of individuals chosen is $p_1 \cup \ldots \cup p_r$.

In the next proposition, we present some properties of the choice rules $C^{\min\max}$ and $C^{\max\min}$, formulating the sense in which they are extremal selections from the paired-admissions choice correspondence.

Proposition 2. *$C^{\min\max}$ and $C^{\max\min}$ have the following properties.*

(1) Let C be any paired-admissions choice rule.

(a) For any $I \subseteq \mathcal{I}$ and $i \in I$ such that $\tau(i) = \emptyset$ or $\tau(i) = \{t_1, t_2\}$,

$$i \in C(I) \implies i \in C^{\min\max}(I).$$

(b) For any $I \subseteq \mathcal{I}$ and $i \in I$ such that $\tau(i) = \{t_1\}$ or $\tau(i) = \{t_2\}$,

$$i \in C^{\min\max}(I) \implies i \in C(I).$$

(c) *For any* $I \subseteq \mathcal{I}$, *the highest priority individual in* $I \setminus C^{\min\max}(I)$ *has a weakly lower priority than the highest priority individual in* $I \setminus C(I)$.

(2) *Let* C *be any paired-admissions choice rule.*

(a) *For any* $I \subseteq \mathcal{I}$ *and* $i \in I$ *such that* $\tau(i) = \{t_1\}$ *or* $\tau(i) = \{t_2\}$,

$$i \in C(I) \implies i \in C^{\max\min}(I).$$

(b) *For any* $I \subseteq \mathcal{I}$ *and* $i \in I$ *such that* $\tau(i) = \emptyset$ *or* $\tau(i) = \{t_1, t_2\}$,

$$i \in C^{\max\min}(I) \implies i \in C(I).$$

(c) *For any* $I \subseteq \mathcal{I}$, *the lowest priority individual in* $C^{\max\min}(I)$ *has a weakly higher priority than the lowest priority individual in* $C(I)$.

The next result shows that certain pairs of individuals are always selected together, thus motivating the term "paired-admissions choice rule."

Proposition 3. *Let* C *and* C' *be two paired-admissions choice rules and* I *be a set of individuals. If either*

(1) i *is the lowest priority individual with no traits in* $C(I)$ *and* i' *is the lowest priority individual with both traits in* $C(I)$ *or*

(2) i *is the lowest priority individual with only trait* t_1 *in* $C(I)$ *and* i' *is the lowest priority individual with only trait* t_2 *in* $C(I)$, *then*

$$i \in C'(I) \iff i' \in C'(I).$$

5. Heterogeneous Jobs

In this section, we extend our analysis to the more general case with multiple jobs. There is a set of individuals $\mathcal{I}$ and a set of jobs $\mathcal{J}$. Each individual $i \in \mathcal{I}$ has a strict preference order $\succ_i$ over the set of jobs and the outside option of remaining unmatched. A job is **acceptable** for an individual if she prefers it to the outside option.

For each job $j \in \mathcal{J}$,

- there are $q_j > 0$ identical positions to allocate,
- there is a strict priority order π_j over the set of individual $\mathcal{I}$, and
- there is a set of (possibly) job-specific traits $\mathcal{T}_j$.

Let

$$\mathcal{T} = \bigcup_{j \in \mathcal{J}} \mathcal{T}_j$$

denote the set of all traits across all jobs.

Each individual $i \in \mathcal{I}$ has a set of traits $\tau(i) \subseteq \mathcal{T}$. For each job $j \in \mathcal{J}$ and trait $t \in \mathcal{T}_j$, let $r_{j,t} \in \mathbb{N}$ be the number of positions are reserved for individuals with trait t.

For any job $j \in \mathcal{J}$, the definition of the horizontal envelope choice rule $C_j^{\boxtimes}$ immediately extends to this framework under the one-to-one reserve matching convention. Similarly, assuming there are only two traits, the definition of the paired-admissions choice correspondence also extends under the one-for-all reserve matching convention.

Over the last fifteen years, the following algorithm Gale and Shapley (1962) has become the mechanisms of choice for priority-based allocation with heterogenous goods, where the policies of the institutions are captured through the choice rules that are used in conjunction with this algorithm.

Agent-Proposing Deferred Acceptance Algorithm (DA)

Step 1: Each individual applies to her most preferred acceptable job if such a job exists. Suppose that I_j^1 is the set of individuals who apply to job j. Job j tentatively accepts individuals in $C_j^{\boxtimes}(I_j^1)$ and permanently rejects the rest. If there are no rejections, then stop.

Step k: Each individual who was rejected in Step $k-1$ applies to her next preferred acceptable job, if such a job exists. Suppose that I_j^k is the union of the set of individuals who were tentatively accepted by job j in Step $k-1$, and the set of individuals who just proposed to job j. Job j tentatively accepts individuals in $C_j^{\boxtimes}(I_j^k)$ and permanently rejects the rest. If there are no rejections, then stop.

Extension of our analysis to the case with heterogeneous jobs through DA is straightforward, provided that the choice rule of each job satisfies the following two conditions.[10]

Definition 5. *(Kelso and Crawford, 1982) A choice rule C satisfies the* ***substitutes*** *condition, if, for every $I \subseteq \mathcal{I}$,*

$$i \in C(I) \text{ and } i' \neq i \implies i \in C(I \setminus \{i'\}).$$

Definition 6. *(Aygün and Sönmez, 2013) A choice rule C satisfies the* ***irrelevance of rejected individuals*** *condition, if, for every $I \subseteq \mathcal{I}$,*

$$i \in I \setminus C(I) \implies C(I \setminus \{i\}) = C(I).$$

As the following two results imply, while a joint implementation of the agent-proposing deferred acceptance algorithm with the horizontal envelope choice rule is straightforward under the one-to-one reserve matching convention, this is not possible for any paired-admissions choice rule under the one-to-all reserve matching convention.

Proposition 4. *The horizontal envelope choice rule $C^{\boxtimes}$ satisfies the substitutes condition and the irrelevance of rejected individuals condition.*

Proposition 5. *There exists no paired-admissions choice rule that satisfies the substitutes condition.*

[10] See for example Hatfield and Milgrom (2005) and Aygün and Sönmez (2013).

6. Applications

In this section, we present three large scale practical applications of our model from Chile, India, and Brazil, and present how our proposed mechanism improves upon the mechanisms of choice in these applications.

6.1. School Inclusion Law in Chile. With the promulgation of the School Inclusion Law in Chile in 2015, a centralized school choice system has been in the process of being adopted in Chile, following a similar series of reforms throughout the world (Correa et al., 2019). The system is the product of an ongoing collaboration between the Chilean Ministry of Education and a team of researchers from economics and operations research, and it covers all grades prior to higher education (i.e., Pre-K to grade 12). The system was first implemented in 2016 as a pilot program in the smallest of the sixteen regions of Chile, and it has been adopted in all regions but the Metropolitan Area of Santiago by 2019, where over 274,000 students applied to more than 6400 schools. The system is expected to be adopted throughout Chile in 2020.

As many of its predecessors, the Chilean school choice system is based on the celebrated deferred acceptance algorithm, and the following three features in its design make it a perfect application of our model:

(1) In order to promote diversity, the School Inclusion Law includes affirmative action policies for financially disadvantaged students and children with special needs. Under the new system, these policies are implemented through reserved seats at each school. In addition, a number of schools are allowed to reserve seats for high-achieving students. Hence, using our terminology there are three traits, *Financially disadvantaged*, *Special needs*, *High-achieving*, where a student potentially can have any subset of these traits, possibly including none of them.[11]
(2) While a student with multiple traits (say a financially disadvantaged student who is also high-achieving) is eligible for reserved seats for each of her traits, she "consumes" only one of the reserved seats upon receiving a seat. This feature in Chilean design eliminates potential complementarities between the regular students and students with multiple traits.
(3) Reserved seats at each school are implemented in the form of a *soft lower bound* (i.e., as a minimum guarantee).

A subtle implication of the second design feature is that it allows the model to be interpreted as an application of the *matching with contracts* model of Hatfield and Milgrom (2005), where the contractual term between a school and a student specifies which of the four types of seats (i.e., open seats, reserved seats for financially disadvantaged students,

[11] Students with none of the three traits are referred as *Regular*.

reserved seats for special needs students, and reserved seats for high-achieving students) the student receives, which is studied in Kurata et al. (2017). However, the theory of matching with contracts is developed under the assumption that students have strict preferences over all their contracts, which in this context corresponds to them having strict preferences on the specific type of seats they receive at each school. Since students have preferences over only schools, a tie-breaking rule is used to construct student preferences over specific type of seats at each school. In Correa et al. (2019), the designers emphasize that the choice of a tie-breaking rule is not straightforward, and it has distributional consequences. In order to implement the reserves in the form of a minimum guarantee, they break ties in a way each student is assumed to prefer reserved seats for any of their traits to open seats. When each student has at most one trait, this construction assures that the reserves are implemented as a minimum guarantee (Hafalir et al. (2013), Sönmez and Yenmez (2019a)).[12] However, as we present in the following two examples, interpreting this problem as an application of matching with contracts and relying on tie-breaking between reserved seats results in undesired outcomes.

Example 3. Suppose there is only one school s with three seats. There are four students i_1, i_2, i_3, i_4 who are priority ranked as follows:

$$i_1 \ \pi_s \ i_2 \ \pi_s \ i_3 \ \pi_s \ i_4.$$

One of the seats is reserved for students with trait financially disadvantaged t_d, and one of the seats is reserved for students with trait high-performing t_h. Student i_1 is a regular student with neither of the traits, whereas student i_2 has both traits, student i_3 has the high-performing trait t_h only, and student i_4 has the financially disadvantaged trait t_d only.

Let s_o denote the open seat, s_d denote the reserve seat for financially disadvantaged students, and s_h denote the reserve seat for high-achieving students. Hence student i_2 receives preferential treatment for both seat s_d and seat s_h, whereas student i_3 receives preferential treatment for seat s_h only, and student i_4 receives preferential treatment for seat s_d only. Hence, student priorities for each seat is given as follows:

$$i_1 \ \pi_{s_o} \ i_2 \ \pi_{s_o} \ i_3 \ \pi_{s_o} \ i_4$$

$$i_2 \ \pi_{s_h} \ i_3 \ \pi_{s_h} \ i_1 \ \pi_{s_h} \ i_4$$

$$i_2 \ \pi_{s_d} \ i_4 \ \pi_{s_d} \ i_1 \ \pi_{s_d} \ i_3.$$

As for the tie-breaking, let us assume reserve seat s_h is preferred to reserve seat s_d for any student who either receives preferential treatment for both reserve seats or for neither of

[12]See Dur et al. (2018) for an unintended consequence of an erroneous choice of a tie-breaking rule by the Boston Public Schools system: The choice of the tie-breaking rule undermined the stated policy objectives of the school district, and resulted in a de facto elimination of walk-zone priority by implementing the walk-zone reserves on a minimum guarantee basis rather than over-and-above basis.

the reserve seats. Since by the Chilean design each student is also assumed to prefer seats at one of her traits to open seats, this results in the following construction of preferences for the students:

$$s_o \succ_{i_1} s_h \succ_{i_1} s_d$$
$$s_h \succ_{i_2} s_d \succ_{i_2} s_o$$
$$s_h \succ_{i_3} s_o \succ_{i_3} s_d$$
$$s_d \succ_{i_4} s_o \succ_{i_4} s_h.$$

Therefore, under the student-proposing deferred acceptance algorithm, at Step 1 student i_1 applies to the open seat s_o, whereas students i_2 and i_3 both apply for the reserved seat s_h, and student i_4 applies to the reserved seat s_d. Reserved seat s_h holds student i_2 and rejects student i_3, whereas the open seat s_o and the reserved seat s_d each hold their only applicant. At Step 2, student i_3 applies to the open seat s_o, only to be rejected again since student i_1 who is on hold for the open seat has higher priority for the open seat. Finally at Step 3, student i_3 applies to the reserve seat s_d, and gets rejected for a third time since student i_4 who is on hold for the reserved seat s_d has higher priority for the seat s_d having the financially disadvantaged trait. This results in the following matching

$$\begin{pmatrix} s_o & s_h & s_d \\ i_1 & i_2 & i_4 \end{pmatrix}$$

of students to seats, and hence the set of students admitted to school s are $\{i_1, i_2, i_4\}$. This outcome is undesired for the following reason: Observe that student i_2 has the highest priority at not only seat s_h but also s_d. Therefore, had she not been artificially assumed to prefer seat s_h to s_d, she could have been instead assigned the seat s_d, which in turn would allow student i_3 to receive seat s_h resulting in the matching,

$$\begin{pmatrix} s_o & s_h & s_d \\ i_1 & i_3 & i_2 \end{pmatrix}.$$

This alternative outcome is preferred to the outcome of the Chilean system, because it replaces the third priority student i_3 with the fourth priority student i_4, while still satisfying both reserves. This outcome is indeed the outcome of the horizontal envelope choice rule, which is agnostic about which type of reserved seat an agent receives when she has multiple traits. □

The Chilean system induces a choice rule C^{Chile} for any given tie-breaking rule, and in Example 3 the choice rule C^{Chile} uses reservations fully. Hence, by Proposition 1 in Section 4, the choice rule $C^{\boxtimes}$ admits higher priority students than the choice rule C^{Chile}

in all examples where the latter choice rule uses reservations fully. But our next example shows that, in general the choice rule C^{Chile} may fail to use reservations fully.

Example 4. Suppose there is only one school s with three seats. There are four students i_1, i_2, i_3, i_4 who are priority ranked as follows:

$$i_1 \ \pi_s \ i_4 \ \pi_s \ i_2 \ \pi_s \ i_3.$$

One of the seats is reserved for students with trait financially disadvantaged t_d, and one of the seats is reserved for students with trait high-performing t_h. Students i_1, i_4 are both regular students with neither of the traits, whereas student i_2 has both traits, and student i_3 has the financially disadvantaged trait t_d only.

Let s_o denote the open seat, s_d denote the reserve seat for financially disadvantaged students, and s_h denote the reserve seat for high-achieving students. Hence, student i_2 receives preferential treatment for both seats s_d and s_h, whereas student i_3 receives preferential treatment for seat s_d only. Hence, student priorities for each seat is given as follows:

$$i_1 \ \pi_{s_o} \ i_4 \ \pi_{s_o} \ i_2 \ \pi_{s_o} \ i_3$$
$$i_2 \ \pi_{s_h} \ i_1 \ \pi_{s_h} \ i_4 \ \pi_{s_h} \ i_3$$
$$i_2 \ \pi_{s_d} \ i_3 \ \pi_{s_d} \ i_1 \ \pi_{s_d} \ i_4.$$

As for the tie-breaking, let us assume reserve seat s_d is preferred to reserve seat s_h for any student who either receives preferential treatment for both reserve seats or for neither of the reserve seats. Since by the Chilean design each student is also assumed to prefer seats at one of her traits to open seats, this results in the following construction of preferences for the students:

$$s_o \ \succ_{i_1} \ s_d \ \succ_{i_1} \ s_h$$
$$s_d \ \succ_{i_2} \ s_h \ \succ_{i_2} \ s_o$$
$$s_d \ \succ_{i_3} \ s_o \ \succ_{i_3} \ s_h$$
$$s_o \ \succ_{i_4} \ s_d \ \succ_{i_4} \ s_h.$$

So under the student-proposing deferred acceptance algorithm, at Step 1 students i_1 and i_4 both apply to the open seat s_o, whereas students i_2 and i_3 both apply for the reserved seat s_d. Open seat s_o holds student i_1 and rejects student i_4, whereas the reserved seat s_d holds student i_2 and rejects student i_3. At Step 2, student i_4 applies to the reserved seat s_d and student i_3 applies to the open seat s_o, and both students are rejected since these seats are each holding higher-priority students. Finally at Step 3, both students i_3 and i_4 apply to the reserved seat s_h, which holds student i_4 and rejects student i_3. Since student i_3 is rejected from all seats, the algorithm terminates at the end of Step 3 finalizing all assignments. This results in the following matching

$$\mu = \begin{pmatrix} s_o & s_h & s_d \\ i_1 & i_4 & i_2 \end{pmatrix}$$

of students to seats, and hence the set of students admitted to school s are $\{i_1, i_4, i_2\}$. This outcome is undesired for the following reason: While it is possible to assign both reserve seats to students who have priority for these seats through an alternative matching of

$$\nu = \begin{pmatrix} s_o & s_h & s_d \\ i_1 & i_2 & i_3 \end{pmatrix},$$

matching μ instead only utilizes the reserve for financially disabled trait, and thus it de facto "converts" the reserve for high-achieving trait to an open position. Hence, the tie-breaking rule results in a situation where student i_2 who has the flexibility to receive either of the two reserved seats to give up this flexibility forcing her to rigidly receive the reserve seat s_d, which in turn means no one else can benefit from the high-achieving reserve. The end result is, only one of the reserve seats is utilized for affirmative action under the choice rule C^{Chile} even though both could have been utilized for this purpose. In contrast, both reserved seats are utilized under the horizontal envelope choice rule, which is agnostic about which type of reserved seat a student receives when she has multiple traits. □

Example 3 shows that the choice rule induced by the Chilean system allows for justified envy, whereas Example 4 shows that it fails to use the reservations fully. In contrast, not only the horizontal envelope choice rule $C^{\boxtimes}$ satisfies these properties, it is also the unique choice rule that satisfies these properties along with the modest efficiency requirement of non-wastefulness.

6.2. Supreme Court of India Horizontal Adjustment Subroutine. Built into the country's constitution, one of the world's most comprehensive affirmative action programs is implemented in India. Allocation of government positions and seats at publicly funded educational institutions have to comply with rules outlined by the landmark Supreme Court judgement *Indra Sawhney and others v. Union of India (1992)*,[13] widely known as the *Mandal Commission Case*. Under these rules, an allocation system that relies on an objective merit list of candidates is integrated with two types of reservations referred as *vertical reservations* and *horizontal reservations*. Of the two types of reservations, vertical reservations are intended as the higher-level provisions, targeted for classes that faced historical discrimination, and implemented on a "set aside" (or equivalently "over-and-above") basis. Horizontal reservations, on the other hand, are intended as lower-level provisions, targeted for other disadvantaged groups (such as women or disabled citizens), and implemented on a

[13] The case is available at `https://indiankanoon.org/doc/1363234/` (last accessed on 11/29/2019).

"minimum guarantee" basis. The distinction between the higher-level provisions role of vertical reservations and the lower-level provisions role of horizontal reservation is justified through the mechanisms of their implementation (Sönmez and Yenmez, 2019a). In the absence of horizontal reservations, implementation of vertical reservations is straightforward: First open (i.e., unreserved) positions are allocated based on merit scores only, and next for each of the (mutually exclusive) groups eligible for vertical reservation, reserved positions are allocated to the remaining members of the group based on their merit scores.

What is less clear is how to implement horizontal reservations to eligible groups (that could potentially overlap) in the form of minimum guarantees. The subroutine for implementing horizontal reservations is first introduced by the Supreme Court judgement *Anil Kumar Gupta v. State of U.P. (1995)*,[14] and further elaborated in the Supreme Court judgement *Rajesh Kumar Daria v. Rajasthan Public Service Commission and others (2007)* as follows:[15]

> If 19 posts are reserved for SCs (of which the quota for women is four), 19 SC candidates shall have to be first listed in accordance with merit, from out of the successful eligible candidates. If such list of 19 candidates contains four SC women candidates, then there is no need to disturb the list by including any further SC women candidate. On the other hand, if the list of 19 SC candidates contains only two woman candidates, then the next two SC woman candidates in accordance with merit, will have to be included in the list and corresponding number of candidates from the bottom of such list shall have to be deleted, so as to ensure that the final 19 selected SC candidates contain four women SC candidates. (But if the list of 19 SC candidates contains more than four women candidates, selected on own merit, all of them will continue in the list and there is no question of deleting the excess women candidate on the ground that 'SC-women' have been selected in excess of the prescribed internal quota of four.)

This procedure, which we refer as *Supreme Court of India horizontal adjustment subroutine* (or in short *SCI horizontal subroutine*) is mandated throughout India for each horizontal trait, and urged to be implemented independently for each of the vertical categories; i.e for the open category, to be followed by each of the vertical categories. This is referred as *compartmentalized horizontal reservation*.

In India the only federally mandated horizontal reservation is for disabled people, and it is known as persons with disability (PwD) reservation.[16] In applications with only one

[14]The case is available at https://indiankanoon.org/doc/1055016/ (last accessed on 11/29/2019).

[15]The case is available at https://indiankanoon.org/doc/698833/ (last accessed on 11/29/2019).

[16]Horizontal PwD reservation is federally mandated by the Supreme Court judgement *Union Of India & Anr vs National Federation Of The Blind & ... on 8 October, 2013*, available at https://indiankanoon.org/doc/178530295/ (last accessed on 11/29/2019).

horizontal trait, integrating SCI horizontal subroutine with vertical reservation is straightforward: The subroutine is to be directly applied to each vertical category.[17]

However, there are multiple horizontal traits in many applications. For example, horizontal reservation for women is mandated in several states including in Bihar with 35%, Andhra Pradesh with $33\frac{1}{3}$%, and Madhya Pradesh, Uttarakhand, Chhattisgarh, Rajasthan, and Sikkim with 30% each. In many applications, there are other horizontal traits as well, such as ex-servicemen, sportsmen, etc. In those applications, the standard procedure is implementing the SCI horizontal subroutine for each of the horizontal traits. Processing multiple horizontal reservations is also straightforward when each individual is either qualified for at most one horizontal trait, or she is forced to declare at most one horizontal trait. In that case, it is immaterial in what order the adjustments are made via SCI horizontal subroutine. Sönmez and Yenmez (2019a) analyze the Supreme Court's allocation procedure for this case where it is well-defined.

What is analytically more challenging, however, is the case when individuals qualify for multiple horizontal reservations. Before analyzing the impact of sequential implementation of the SCI horizontal subroutine for each trait, it is important to clarify one important aspect of this version of the problem: Suppose there are two horizontal traits, one for women and another for persons with disability, and consider a female candidate with a disability who is admitted. A key policy question is, whether this candidate counts for both of the horizontal reservations or only one of the horizontal reservations. To the best of our knowledge, this critical question is not addressed by any of the court cases. Hence as far as we can tell, both interpretations are consistent with Indra Sawhney (1992) and the subsequent Supreme Court rulings. However, we believe the more widespread interpretation is the case where each candidate counts for only one of the horizontal traits. That is because, in most applications in India, the number of positions are explicitly given for each vertical category-horizontal trait pair, and as a consequence individuals are assigned to positions attached to category-trait pairs. For example, when positions are announced for the classifications Open Category-Women and Open Category-PwD, a disabled woman can be assigned a position from either classification (or simply the Open Category without using either one of her horizontal traits). Hence, most of the applications are consistent

[17]Even with only one horizontal trait, however, there is one subtle aspect of horizontal reservation (Sönmez and Yenmez, 2019a). Under Indra Sawhney (1992), while all candidates are eligible for open positions (without any horizontal adjustments), only members of General Category (i.e., candidates who do not qualify for vertical reservation) are eligible for horizontal adjustment for open positions. Horizontal adjustments for other candidates is to be carried out in their respective vertical categories. While this restriction does not make application of the SCI horizontal subroutine any more complicated, it introduces a number of shortcomings in the resulting choice rule. See Sönmez and Yenmez (2019a) numerous lawsuits due to these shortcomings, as well as a recommended resolution.

with individuals "using up" a position from only one of the horizontal reservations. Adhering to this interpretation of horizontal reservation, we next present two shortcomings of the practice of sequentially implementing the SCI horizontal adjustment subroutine.[18]

Example 5. Suppose there is a job with three (open-category) positions, where one position is horizontally reserved for women and one position is horizontally reserved for persons with disability. Let us denote the horizontal trait of being woman as (W) and the horizontal trait of being disabled as (D).

There are three male candidates i_1, i_2, i_3, and two female candidates i_4, i_5. Only one of the candidates, the female candidate i_4 is disabled. Candidates are merit ranked as follows:

$$i_1 \; \pi \; i_2 \; \pi \; i_3 \; \pi \; i_4 \; \pi \; i_5.$$

Prior to horizontal adjustments, the three highest merit ranking candidates i_1, i_2, i_3, are tentatively assigned to available positions. Since neither the women reservation nor the disability reservation is satisfied, adjustments are needed for both horizontal traits.

If, the SCI horizontal adjustment subroutine is used in the sequence D-W, then

(1) first the highest priority unmatched disabled candidate i_4 receives the horizontally reserved position for disabled at the expense of the lowest merit ranking non-diabled candidate i_3 who was on hold, and
(2) subsequently the highest priority unmatched female candidate i_5 receives the horizontally reserved position for women at the expense of the lowest merit ranking male candidate i_2 who was on hold,

resulting in the admission of the set of individuals $\{i_1, i_4, i_5\}$.

If, on the other hand, the SCI horizontal adjustment subroutine is used in the sequence W-D, then

(1) first the highest priority unmatched female candidate i_4 receives the horizontally reserved position for women at the expense of the lowest merit ranking male candidate i_3 who was on hold, and
(2) next due to the lack of disabled candidates among unmatched candidates, no additional adjustment can be made

resulting in the admission of the set of individuals $\{i_1, i_2, i_4\}$. □

Example 3 highlights two shortcomings of sequential implementation of SCI horizontal adjustment subroutine: First, the outcome of this process depends on the order the horizontal traits, paving the way for favoritism. This means, unless the process order is explicitly

[18]See Example 1 in Sönmez and Yenmez (2019a) for a shortcoming of the SCI horizontal adjustment subroutine under the alternative interpretation where an individual uses up a position from each of the horizontal traits she has upon admission.

stated, the resulting mechanism is not mathematically well-defined. Perhaps the more important shortcoming is the second one. When the traits are processed in the sequence W-D, only one of the horizontally reserved positions are filled with candidates who are eligible for these positions, even though it is feasible to assign both of these positions who are eligible for horizontal reservation. Processing of trait W prior to trait D results in a "mismatch" of the horizontally reserved W position to the disabled female candidate i_4, who could have as well received the horizontally reserved D position. Therefore, the resulting choice rule fails to use the reservations fully.

Observe that adjustment for horizontal traits is a direct application of our model, and both shortcomings can be avoided by replacing sequential implementation of the SCI horizontal adjustment subroutine with the horizontal envelope choice rule $C^{\boxtimes}$ whenever there are overlapping horizontal traits.

6.3. The Law of Social Quotas in Brazil. On August 29, 2012, President Dilma Rousseff enacted the *Law of Social Quotas* in Brazil, which requires public colleges to reserve at least half of their seats for graduates of public high schools. The law also requires that, within seats reserved for graduates of public high schools, half should be reserved for low income families. Finally, the law also requires that, again within seats reserved for graduates of public high schools, a percentage equal to their share in the State population should be reserved for those who declare themselves as black, mixed, or indigenous. Thus, for our application in Brazil, there are three traits p, ℓ, and m, where

- trait p indicates graduating from a public school,
- trait ℓ indicates being low income, and
- trait m indicates being of black, mixed or indigenous descent.

By the Law of Social Quotas, there are reserved seats for

(1) students who have trait p,
(2) students who have traits p and ℓ, and
(3) students who have traits p and m.

Let r_p denote the number seats reserved for graduates of public schools, r_ℓ the number of seats reserved for graduates of public schools who are low income, and r_m the number of seats reserved for graduates of public schools who are of black, mixed or indigenous descent. Since the latter two reserves are within the reserves for public school graduates, we have $r_\ell \leq r_p$ and $r_m \leq r_p$.

Observe that, the traits are overlapping. Moreover, the traits ℓ and m are useful only if an applicant also has the trait p. Therefore, even though there are three traits in this application, there are complementarities between only two groups of individuals, namely individuals with trait p only, and individuals with all three traits. While our results in

Section 4 do not immediately apply to the application for Brazil, we can build on these results to design a class of choice rules since complementarities are restricted to two groups for this application as well.

Even though the Law of Social Quotas requires reserved seats for students with sets of trait $\{p\}$, $\{p, \ell\}$, and $\{p, m\}$, it does not provide any choice rule to implement these reserves. Due to the challenges of implementing overlapping reserves, several universities have adopted choice rules that rely on dividing the reserves given by the law into finer reserves for each combination of traits. Under these rules, there are (non-overlapping) reserves for

(1) public school graduates who are low income and of black, mixed or indigenous descent,
(2) public school graduates who are low income but not of black, mixed or indigenous descent,
(3) public school graduates who are of black, mixed or indigenous descent but not low income, and
(4) public school graduates who are neither low income, nor of black, mixed or indigenous descent.

Aygün and Bó (2016) analyze several of these choice rules, presenting their shortcomings and correcting their shortcomings. However their analysis also rely on the finer division of the reserves that introduces some artificial restrictions in addition to the restrictions that are imposed by the law. In contrast, in our design we maintain the overlapping reserves structure given in the Law of Social Quotas, and directly implement the provisions in the law without imposing any additional restrictions.

The following choice rule for admissions of students in Brazil can be used in conjunction with any paired-admissions choice rule C where C is used to allocate public school seats first and the two traits are minority m and low income ℓ. Let I be a set of applicants.

Horizontal Choice Rule $\mathbf{C^{hor} \circ C}$

Step 1: Choose $C(I)$ for the seats reserved for students graduating from public schools.

Step 2: Choose remaining individuals with the highest priorities for the remaining seats.

The set of individuals chosen is denoted by $(C^{hor} \circ C)(I)$. In particular, the two extremal paired-admissions choice rules can be used in this construction.

Proposition 6. *$C^{hor} \circ C^{\min\max}$ and $C^{hor} \circ C^{\max\min}$ have the following properties.*

(1) Let C be any paired-admissions choice rule.

(a) *For any* $I \subseteq \mathcal{I}$ *and* $i \in I$ *such that* $\tau(i) = \{p, \ell, m\}$,

$$i \in (C^{hor} \circ C)(I) \implies i \in (C^{hor} \circ C^{\min\max})(I).$$

(b) *For any* $I \subseteq \mathcal{I}$ *and* $i \in I$ *such that* $\tau(i) = \{p, \ell\}$ *or* $\tau(i) = \{p, m\}$,

$$i \in (C^{hor} \circ C^{\min\max})(I) \implies i \in (C^{hor} \circ C)(I).$$

(2) *Let C be any paired-admissions choice rule.*

(a) *For any* $I \subseteq \mathcal{I}$ *and* $i \in I$ *such that* $\tau(i) = \{p, \ell\}$ *or* $\tau(i) = \{p, m\}$,

$$i \in (C^{hor} \circ C)(I) \implies i \in (C^{hor} \circ C^{\max\min})(I).$$

(b) *For any* $I \subseteq \mathcal{I}$ *and* $i \in I$ *such that* $\tau(i) = \{p, \ell, m\}$,

$$i \in (C^{hor} \circ C^{\max\min})(I) \implies i \in (C^{hor} \circ C)(I).$$

(c) *For any* $I \subseteq \mathcal{I}$, *the lowest priority individual in* $(C^{hor} \circ C^{\max\min})(I)$ *has a weakly higher priority than the lowest priority individual in* $(C^{hor} \circ C)(I)$.

Remark 1. For an individual i such that $\tau(i) = \emptyset$ and $\tau(i) = \{p\}$, there is no preference comparison between $C^{hor} \circ C^{\min\max}$ and $C^{hor} \circ C^{\max\min}$. For a set of individuals, the highest priority rejected individual by $C^{hor} \circ C^{\min\max}$ can have a strictly lower or strictly higher priority than the highest priority rejected individual by $C^{hor} \circ C^{\max\min}$.

7. Conclusion

We have presented a theory of overlapping reserves both for the case of one-to-one reserve matching convention and also for the case of one-to-all reserve matching convention when there are complementarities between two groups only. There is a unique natural choice rule that emerges under the first convention, the horizontal envelope choice rule, and, hence, if there is any flexibility to select one of the conventions we believe the case for the one-to-one reserve matching is much stronger. Our results have direct policy implications for a variety of real-life allocation problems, including school choice in Chile, public job allocation in India, and college admissions in Brazil.

References

Abdulkadiroğlu, Atila, "College admissions with affirmative action," *International Journal of Game Theory*, 2005, *33* (4), 535–549.

——— **and Tayfun Sönmez**, "School choice: A mechanism design approach," *American Economic Review*, June 2003, *93* (3), 729–747.

———**, Parag A. Pathak, Alvin E. Roth, and Tayfun Sönmez**, "The Boston Public School Match," *American Economic Review*, May 2005, *95* (2), 368–371.

______, ______, and ______, "The New York City High School Match," *American Economic Review*, 2005, *95* (2), 364–367.

Alva, Samson, "Stability and complementarities in matching," 2019. Working paper.

Aygün, Orhan and Bertan Turhan, "Dynamic Reserves in Matching Markets," *Working paper*, 2016.

______ **and Inácio Bó**, "College admission with multidimensional privileges: The Brazilian affirmative action case," *Working paper, WZB Berlin Social Science Center.[869]*, 2016.

Aygün, Orhan and Tayfun Sönmez, "Matching with Contracts: Comment," *American Economic Review*, 2013, *103* (5), 2050–2051.

Azevedo, Eduardo M. and Jacob D. Leshno, "A Supply and Demand Framework for Two-Sided Matching Markets," *Journal of Political Economy*, 2016, *124* (5), 1235–1268.

Balinski, Michel and Tayfun Sönmez, "A Tale of Two Mechanisms: Student Placement," *Journal of Economic Theory*, January 1999, *84* (1), 73–94.

Baswana, Surender, Partha Pratim Chakrabarti, Sharat Chandran, Yashodhan Kanoria, and Utkarsh Patange, "Centralized admissions for engineering colleges in India," 2018. Forthcoming in INFORMS Journal on Applied Analytics.

Biro, P., T. Fleiner, R.W. Irving, and D.F. Manlove, "The College Admissions Problem with Lower and Common Quotas," *Theoretical Computer Science*, 2010, *411* (34-36), 3136–3153.

Bó, Inácio, "Fair implementation of diversity in school choice," *Games and Economic Behavior*, 2016, *97*, 54–63.

Budish, Eric, Yeon-Koo Che, Fuhito Kojima, and Paul R. Milgrom, "Designing Random Allocation Mechanisms: Theory and Applications," *American Economic Review*, 2013, *103* (2), 585–623.

Chambers, Christopher P. and M. Bumin Yenmez, "A simple characterization of responsive choice," *Games and Economic Behavior*, 2018, *111*, 217 – 221.

Che, Yeon-Koo, Jinwoo Kim, and Fuhito Kojima, "Stable Matching in Large Economies," *Econometrica*, 2019, *87* (1), 65–110.

Correa, Jose, Rafael Epstein, Juan Escobar, Ignacio Rios, Bastian Bahamondes, Carlos Bonet, Natalie Epstein, Nicolas Aramayo, Martin Castillo, Andres Cristi, and Boris Epstein, "School Choice in Chile," in "Proceedings of the 2019 ACM Conference on Economics and Computation" EC '19 ACM New York, NY, USA 2019, pp. 325–343.

Delacrétaz, David, Scott Duke Kominers, and Alexander Teytelboym, "Refugee Resettlement," 2016. Working Paper.

Doğan, Battal, "Responsive affirmative action in school choice," *Journal of Economic Theory*, 2016, *165*, 69–105.

Doğan, Battal, Serhat Doğan, and Kemal Yıldız, "Lexicographic Choice Under Variable Capacity Constraints," 2019.

Dur, Umut Mert, Parag Pathak, and Tayfun Sönmez, "Explicit vs. Statistical Targeting in Affirmative Action: Theory and Evidence from Chicago's Exam Schools," *Forthcoming in Journal of Economic Theory*, 2016.

Dur, Umut, Scott Duke Kominers, Parag A. Pathak, and Tayfun Sönmez, "Reserve Design: Unintended Consequences and the Demise of Boston's Walk Zones," *Journal of Political Economy*, 2018, *126* (6), 2457–2479.

Echenique, Federico and M. Bumin Yenmez, "How to Control Controlled School Choice," *American Economic Review*, August 2015, *105* (8), 2679–2694.

Ehlers, Lars, Isa E. Hafalir, M. Bumin Yenmez, and Muhammed A. Yildirim, "School choice with controlled choice constraints: Hard bounds versus soft bounds," *Journal of Economic Theory*, 2014, *153*, 648–683.

Fragiadakis, Daniel and Peter Troyan, "Improving matching under hard distributional constraints," *Theoretical Economics*, 2017, *12* (2), 863–908.

Gale, David and Lloyd S. Shapley, "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, January 1962, *69* (1), 9–15.

Goto, Masahiro, Fuhito Kojima, Ryoji Kurata, Akihisa Tamura, and Makoto Yokoo, "Designing Matching Mechanisms under General Distributional Constraints," *American Economic Journal: Microeconomics*, May 2017, *9* (2), 226–62.

Hafalir, Isa E., M. Bumin Yenmez, and Muhammed A. Yildirim, "Effective affirmative action in school choice," *Theoretical Economics*, May 2013, *8* (2), 325–363.

Hafalir, Isa, Fuhito Kojima, and M. Bumin Yenmez, "Interdistrict School Choice: A Theory of Student Assignment," December 2018. Working paper.

Hatfield, John William and Paul R. Milgrom, "Matching with Contracts," *American Economic Review*, September 2005, *95* (4), 913–935.

Htun, Mala, "Is Gender like Ethnicity? The Political Representation of Identity Groups," *Perspectives on Politics*, 2004, *2* (3), 439–458.

Hughes, Melanie M., "Intersectionality, Quotas, and Minority Women's Political Representation Worldwide," *American Political Science Review*, 2011, *105* (3), 604–620.

Kamada, Yuichiro and Fuhito Kojima, "Efficient Matching under Distributional Constraints: Theory and Applications," *American Economic Review*, 2015, *105* (1), 67–99.

______ **and** ______ , "Stability concepts in matching with distributional constraints," *Journal of Economic theory*, 2017, *168*, 107–142.

______ **and** ______ , "Stability and strategy-proofness for matching with constraints: a necessary and sufficient condition," *Theoretical Economics*, 2018, *13* (2), 761–794.

Kelso, Alexander S. and Vincent P. Crawford, "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 1982, *50*, 1483–1504.

Kojima, Fuhito, "School choice: Impossibilities for affirmative action," *Games and Economic*

Behavior, 2012, *75* (2), 685–693.

———, **Akihisa Tamura, and Makoto Yokoo**, "Designing matching mechanisms under constraints: An approach from discrete convex analysis," *Journal of Economic Theory*, 2018, *176*, 803 – 833.

——— **and Parag A. Pathak**, "Incentives and Stability in Large Two-Sided Matching Markets," *American Economic Review*, June 2009, *99* (3), 608–27.

Kominers, Scott Duke and Tayfun Sönmez, "Matching with slot-specific priorities: Theory," *Theoretical Economics*, 2016, *11* (2), 683–710.

Kurata, Ryoji, Naoto Hamada, Atsushi Iwasaki, and Makoto Yokoo, "Controlled school choice with soft bounds and overlapping types," *Journal of Artificial Intelligence Research*, 2017, *58*, 153–184.

Lawler, Eugene, *Combinatorial optimization: networks and matroids*, Dover Publications, 2001.

Ostrovsky, Michael, "Stability in Supply Chain Networks," *American Economic Review*, June 2008, *98* (3), 897–923.

Schummer, James and Azar Abizada, "Incentives in landing slot problems," *Journal of Economic Theory*, 2017, *170*, 29–55.

Sönmez, Tayfun and M. Bumin Yenmez, "Affirmative Action in India via Vertical and Horizontal Reservations," Boston College Working Papers in Economics 977, Boston College Department of Economics March 2019.

——— **and** ———, "Constitutional Implementation of Vertical and Horizontal Reservations in India: A Unified Mechanism for Civil Service Allocation and College Admissions," Boston College Working Papers in Economics 978, Boston College Department of Economics April 2019.

Westkamp, Alexander, "An analysis of the German university admissions system," *Economic Theory*, 2013, *53* (3), 561–589.

FOR ONLINE PUBLICATION ONLY

Appendix A. Mathematical Preliminaries

Let G be the reservation graph of a reservation market. The vertices of G are the individuals in $\mathcal{I}$ and reserved positions. There exists an edge between an individual i and a position reserved for trait t if i has trait t. A **matching** is a set of edges without common vertices. A matching **covers** a vertex if it has an edge adjacent to that vertex.

Lemma 1 (Mendelson-Dulmage Theorem)**.** *Consider a reservation graph. Suppose that there exist a matching that covers individuals in I and a matching that covers reserved positions in S. Then there exists a matching that covers both I and S.*

See Theorem 4.1 in Lawler (2001, Page 191) for a proof of this lemma.

An **alternating path** between matching M_1 and matching M_2 is a path of connected edges that starts at a vertex covered by M_1 but not by M_2 and ends at a vertex covered by M_2 but not by M_1 such that edges of the path belong alternately to M_1 and M_2.

Lemma 2 (Alternating Path)**.** *Let M_1 and M_2 be two distinct matchings that cover the same set of positions in a reservation graph. Suppose that there exists a vertex i covered by M_1 but not by M_2. Then there exists an alternating path between matching M_1 and matching M_2 that starts at i.*

Proof. Let $i_1 \equiv i$ and (i_1, s_1) be the edge that covers i_1 in M_1. Since M_1 and M_2 cover the same set of positions, there exists an edge (i_2, s_1) in M_2. If i_2 is not covered by M_1, then we are done. Otherwise, i_2 is covered by both M_1 and M_2. Let (i_2, s_2) be the edge in M_1 that covers i_2. Since M_1 and M_2 cover the same set of positions, there exists an edge (i_3, s_2) in M_2. If i_3 is not covered by M_1, then we are done. Otherwise, i_3 is covered by both M_1 and M_2. Continue this construction. Since there exists a finite number of vertices, this construction ends in finite time at a vertex i_k covered by M_2 but not by M_1. This finishes the construction of an alternating path starting at i. See Figure 6 for an illustration of the alternating path that is constructed. $\square$

Appendix B. Proofs

In this section, we provide the omitted proofs.

Proof of Proposition 1. Let $I \subseteq \mathcal{I}$ be a set of individuals. To show part (1), note that $|C^{\boxtimes}(I)| = \min\{q, |I|\}$. Furthermore, for choice rule C, $C(I) \subseteq I$ and $|C(I)| \leq q$. Therefore,

$$|C(I)| \leq \min\{q, |I|\} = |C^{\boxtimes}(I)|.$$

We show part (2) by mathematical induction on parameters $(q, (r_t)_{t \subseteq \mathcal{T}})$. We show the claim that for an ordering of agents in $C^{\boxtimes}(I) \setminus C(I)$ and $C(I) \setminus C^{\boxtimes}(I)$ that the k-th agent in

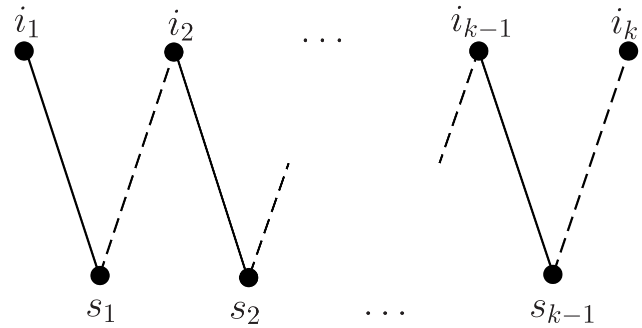

FIGURE 6. The alternating path between M_1 and M_2 constructed in the proof of Lemma 2. The edges in M_1 are solid and the edges in M_2 are dashed.

$C^{\boxtimes}(I) \setminus C(I)$ has a higher priority than the k-th agent in $C(I) \setminus C^{\boxtimes}(I)$, which implies part 2.

For the base case when there are no reserved positions, statement (2) holds because $C^{\boxtimes}$ chooses individuals with the highest priority at Step 2. Now suppose that the claim holds for all parameters bounded above by $(q, (r_t)_{t\subseteq\mathcal{T}})$. Consider parameters $(q, (r_t)_{t\subseteq\mathcal{T}})$. If all individuals in $C^{\boxtimes}(I) \setminus C(I)$ are chosen at Step 2, then the claim holds as in the base case because individuals in $C(I) \setminus C^{\boxtimes}(I)$ are available at Step 2 in the construction of $C^{\boxtimes}(I)$.

Consider the situation when there exists at least one individual in $C^{\boxtimes}(I) \setminus C(I)$ chosen at Step 1. Let i be the individual with the highest priority in $C^{\boxtimes}(I) \setminus C(I)$ chosen at Step 1 and t be the trait of the position that she is matched with. By Lemma 3, $C^{\boxtimes}$ maximally complies with reservations, so in the reservation graph, there exists a matching M_1 that matches $C^{\boxtimes}(I)$ to a set of reserved positions S with maximum cardinality $n(I)$. Since C also maximally complies with reservations, by Mendelson-Dulmage Theorem (see Lemma 1) there exists another matching M_2 that matches $C(I)$ to S both of which have cardinality $n(I)$. By Lemma 2, there exists an alternating path that starts at i and ends at an individual $i' \in C(I) \setminus C^{\boxtimes}(I)$. Therefore, individual i can be replaced with individual i' in $C^{\boxtimes}(I)$ without changing the set of positions covered in the reservation graph for I. Hence, by construction of $C^{\boxtimes}(I)$, $i \; \pi \; i'$ because i' is available when i is chosen at Step 1.

Now consider the reduced market when capacity q and reservation for trait t are both reduced by one and the set of individuals is $I \setminus \{i, i'\}$. In this reduced market, $C^{\boxtimes}(I \setminus \{i, i'\})$ is equal to $C^{\boxtimes}(I) \setminus \{i\}$ for the original market because $i' \notin C^{\boxtimes}(I)$ and the construction of $C^{\boxtimes}(I \setminus \{i, i'\})$ chooses individuals in the same order as they are chosen in $C^{\boxtimes}(I)$. In particular, the set of individuals chosen before i at $C^{\boxtimes}(I)$ are chosen in the same order in $C^{\boxtimes}(I \setminus \{i, i'\})$. Furthermore, after i is chosen the set of updated parameters are exactly the same. Therefore, the same set of individuals are chosen in the same order after i is chosen in $C^{\boxtimes}(I)$. In addition, $C(I) \setminus \{i'\}$ maximally complies with reservations and $i \notin C(I) \setminus \{i'\}$.

By the induction hypothesis, the individuals in $C^{\boxtimes}(I \setminus \{i, i'\})$ and $C(I) \setminus \{i'\}$ can be ordered with the required property, which implies the hypothesis. Therefore, the hypothesis holds for every set of parameters $(q, (r_t)_{t \subseteq \mathcal{T}})$. □

Proof of Theorem 1. We first show that $C^{\boxtimes}$ satisfies the stated properties in several lemmas and then show that the unique choice rule satisfying these properties is $C^{\boxtimes}$.

Lemma 3. *$C^{\boxtimes}$ maximally complies with reservations.*

Proof. Suppose, for contradiction, that $C^{\boxtimes}$ does not maximally comply with reservations. Hence, there exists $I \subseteq \mathcal{I}$ such that $C^{\boxtimes}(I)$ does not maximally comply with reservations for I. Therefore, in the reservation graph for $C^{\boxtimes}(I)$, the maximum cardinality that can be attained by a matching is strictly less than $n(I)$. Let $\bar{I} \subseteq C^{\boxtimes}(I)$ be the set of individuals who are matched to reserved positions in a maximum matching for the reservation graph for $C^{\boxtimes}(I)$. By construction, $|\bar{I}| < n(I)$. Now consider a maximum matching for the reservation graph for I. Let S be the set of positions matched, so $|S| = n(I)$. By Mendelson-Dulmage Theorem (see Lemma 1), there exists a matching that assigns every individual in $\bar{I}$ and every reserved position in S in the reservation graph of I. But this is a contradiction to the construction of $C^{\boxtimes}(I)$ as this choice rule finds a maximal matching in the reservation graph of I, which implies that there cannot exist another matching that assigns a set of individuals that is a proper superset of $C^{\boxtimes}(I)$ to positions in the reservation graph of I. ■

Lemma 4. *$C^{\boxtimes}$ eliminates justified envy.*

Proof. Suppose, for contradiction, that $C^{\boxtimes}$ does not eliminate justified envy. Therefore, there exist a set of individuals $I \subseteq \mathcal{I}$, individuals $i \in C^{\boxtimes}(I)$, $i' \in I \setminus C^{\boxtimes}(I)$ with $n\left((C^{\boxtimes}(I) \setminus \{i\}) \cup \{i'\}\right) \geq n(C^{\boxtimes}(I))$, and $i' \, \pi \, i$. Consider choice rule C such that

$$C(I') = \begin{cases} C^{\boxtimes}(I'), & \text{if } I' \neq I \\ (C^{\boxtimes}(I) \setminus \{i\}) \cup \{i'\}, & \text{if } I' = I. \end{cases}$$

Since $C^{\boxtimes}$ maximally complies with reservations, $C(I')$ maximally complies with reservations for I' whenever $I' \neq I$ because $C(I') = C^{\boxtimes}(I')$. Furthermore,

$$n(C(I)) = n\left((C^{\boxtimes}(I) \setminus \{i\}) \cup \{i'\}\right) \geq n(C^{\boxtimes}(I))$$

and the fact that $C^{\boxtimes}$ maximally complies with reservations by Lemma 3 (i.e., $n(C^{\boxtimes}(I)) = n(I)$) imply that $n(C(I)) = n(I)$ because of the fact that $n(I)$ is the maximum cardinality. Hence, $C(I)$ maximally complies with reservations for I. By Proposition 1, for every $k \leq |C(I)|$, the individual with the k-th highest priority in $C^{\boxtimes}(I)$ has a weakly higher priority than the individual with the k-th highest priority in $C(I)$. This is a contradiction to the construction of C because

$$C(I) = (C^{\boxtimes}(I) \setminus \{i\}) \cup \{i'\}$$

and $i' \; \pi \; i$. ■

Lemma 5. *$C^{\boxtimes}$ is non-wasteful.*

Proof. $C^{\boxtimes}$ is non-wasteful because at the second step all the unfilled positions are filled with the remaining individuals until all positions are filled or all individuals are assigned to positions. ■

Lemma 6. *If a choice rule maximally complies with reservations, eliminates justified envy, and is non-wasteful, then it has to be $C^{\boxtimes}$.*

Proof. Let C be a choice rule that maximally complies with reservations, eliminates justified envy, and is non-wasteful. Suppose, for contradiction, that $C \neq C^{\boxtimes}$. Therefore, there exists $I \subseteq \mathcal{I}$ such that $C(I) \neq C^{\boxtimes}(I)$. Since both choice rules are non-wasteful

$$|C(I)| = |C^{\boxtimes}(I)|.$$

Since $C(I) \neq C^{\boxtimes}(I)$, this equation implies that

$$\left|C^{\boxtimes}(I) \setminus C(I)\right| = \left|C(I) \setminus C^{\boxtimes}(I)\right| > 0.$$

We consider two cases depending on the value of $n(I)$.

Case 1: If $n(I) = 0$, then no individual in I has a trait that has a positive reservation. Therefore, $C^{\boxtimes}(I)$ consists of $\min\{|I|, q\}$ individuals with the highest priority in I. This is a contradiction to the assumption that $C(I)$ eliminates justified envy because any individual $i \in C^{\boxtimes}(I) \setminus C(I) \neq \emptyset$ has a higher priority than any individual $i' \in C(I) \setminus C^{\boxtimes}(I) \neq \emptyset$ and $n((C(I) \setminus \{i'\}) \cup \{i\}) \geq n(C(I)) = 0$. Hence, there is an instance of justified envy for $C(I)$ involving $i \in I \setminus C(I)$ and $i' \in C(I)$, which is a contradiction.

Case 2: Let $n(I) = n > 0$. Therefore, there are n individuals chosen at Step 1 of $C^{\boxtimes}(I)$. For $1 \leq k \leq n$, let i_k be the k-th individual chosen at Step 1 of $C^{\boxtimes}(I)$. Consider a maximum matching M_1 in the reservation graph for $C^{\boxtimes}(I)$ that matches $I_1 \equiv \{i_1, \ldots, i_n\}$. We show that $C(I) \supseteq I_1$. Let S be the set of positions that are matched in M_1. Since C maximally complies with reservations, there exists a maximum matching in the reservation graph for $C(I)$ that has cardinality n. Furthermore, by Mendelson-Dulmage Theorem (see Lemma 1), there exists a matching of a subset of $C(I)$ to positions in S. Let $I_2 \subseteq C(I)$ be the set of these individuals and M_2 be this matching. Suppose, for contradiction, that $I_1 \setminus C(I) \neq \emptyset$. Let i_k be the individual with the lowest index in $I_1 \setminus C(I)$. By Lemma 2, there exists an alternating path between M_1 and M_2 that starts at i_k and ends at a vertex i' covered by M_2 but not by M_1. Therefore, i_k and i' can be replaced with each other in both M_1 and M_2 without decreasing the maximum cardinality. By construction of $C^{\boxtimes}$, $i_k \; \pi \; i'$ because i' is available when i_k is chosen. But this is a contradiction to the assumption that $C(I)$ eliminates justified envy because $i' \in C(I)$, $i_k \in I \setminus C(I)$, $i_k \; \pi \; i'$, and $n(C(I)) = n((C(I) \setminus \{i'\}) \cup \{i_k\})$. Therefore, $I_1 \subseteq C(I)$.

By construction of $C^{\boxtimes}(I)$, every individual in $C^{\boxtimes}(I) \setminus I_1$ is chosen at Step 2. Therefore, these individuals have a higher priority ranking than any individual in $I \setminus C^{\boxtimes}(I)$. Let $i' \in C(I) \setminus C^{\boxtimes}(I)$, which is non-empty by assumption. Therefore, $i' \in I \setminus C^{\boxtimes}(I)$, which means that any individual $i \in C^{\boxtimes}(I) \setminus C(I)$ has a strictly higher priority than i'. This is a contradiction to the assumption that $C(I)$ eliminates justified envy because $i' \in C(I)$, $i \in I \setminus C(I)$, $i \; \pi \; i'$, and $n(C(I)) = n = n((C(I) \setminus \{i'\}) \cup \{i\})$ where the last equation follows from $I_1 \subseteq (C(I) \setminus \{i'\}) \cup \{i\}$ and the fact that $n(I_1) = n$. ■

This finishes the proofs of all the statements. □

Proof of Theorem 2. We provide a series of lemmas to show each claim separately. Recall that for any Step k, $\Delta(k) = r_{t_1}(k) + r_{t_2}(k) - q(k)$.

Lemma 7. *A paired-admission choice rule is non-wasteful.*

Proof. A paired-admission choice rule is non-wasteful because individuals are chosen until all positions are filled or all individuals are chosen. To be more explicit, in the individual-admissions phase, at every step, an individual is chosen as long as there are remaining individuals and positions, and, furthermore, in the paired-admissions phase all remaining positions are filled. ■

Lemma 8. *A paired-admission choice rule maximally complies with reservations.*

Proof. Let C be a paired-admission choice rule. Suppose, for contradiction, that it does not maximally comply with reservations. Then there exists a set of individuals I such that $C(I)$ does not have the maximum cardinality of $n(I)$ in the reservation graph. Therefore, at least one reservation for one of the traits, say t_1 without loss of generality, is not filled by $C(I)$ and there exists an individual in $I \setminus C(I)$ who has t_1. Therefore, the last Step K of the choice rule must be in the individual-admission phase, because in the paired-admission phase all the remaining reservations are filled. Furthermore, $\Delta(K) > 0$ because $q(K) = 1$ and $r_{t_1}(K) + r_{t_2}(K) \geq 2$ since $r_{t_1}(K) \geq 1$ and $r_{t_1}(K) + r_{t_2}(K) = 1$ implies that $r_{t_1}(K) = 1$ and $r_{t_2}(K) = 0$. But if this were the case then at Step K, Case a would holds and an individual with trait t_1 would have been selected at Step K since an individual with trait t_1 is available and therefore all the reserved positions would have been filled.

In addition, whenever $\Delta(k) > 0$, the choice rule selects the highest priority remaining individual who has the maximum number of traits for which reservations are not filled among the remaining individuals ($\Delta(k) > 0$ can only happen for Case a and Case b.a). If $\Delta(k) > 0$ for every Step k, we get a contradiction to the assumption that $C(I)$ does not maximally comply with reservations for I. Therefore, there exists a last Step k' at which $\Delta(k') = 0$ because, for every Step k of the individual-admissions phase,

$$\Delta(k+1) - \Delta(k) = -\delta(i|I \setminus I(k)) \in \{1, 0, -1\},$$

where i is the individual chosen at Step k. Hence, $\Delta(k+1) - \Delta(k)$ can change at most by one at every step. Since this difference goes up by one after Step k', the individual who was chosen does not have a trait for which there were positive reservations. However, $r_{t_1}(k') > 0$ since $r_{t_1}(K) > 0$ and there exists at least one unchosen individual with trait t_1. At Step k', we choose an individual either by Case a or one of the subcases of Case b.c. since $\Delta(k') = 0$ and we are in the individual-admissions phase. Therefore, an individual with a trait for which there are positive remaining reservations must be chosen at Step k'. We get a contradiction. ■

Lemma 9. *A paired-admission choice rule eliminates justified envy.*

Proof. Let C be a paired-admission choice rule. Suppose, for contradiction, that it does not eliminate justified envy. Then, there exist $I \subseteq \mathcal{I}$, $i \in C(I)$, and $i' \in I \setminus C(I)$ such that $i' \; \pi \; i$ and $n\left((C(I) \setminus \{i\}) \cup \{i'\}\right) \geq n(C(I))$. Since i is chosen and i' is rejected even though i' has a higher priority than i, it must be the case that i has a trait that i' does not because of the construction of $C(I)$. Without loss of generality, let t_1 be this trait. We consider two cases depending on whether i has trait t_2 or not.

Case 1: Consider the case when i has trait t_2, so $\tau(i) = \{t_1, t_2\}$. Since $n\left((C(I) \setminus \{i\}) \cup \{i'\}\right) \geq n(C(I))$, there must be at least $r_{t_1} + 1$ individuals in $C(I)$ who have trait t_1. Therefore, the paired-admission choice correspondence must have terminated at the individual-admissions phase because, otherwise, if it ends at the paired-admissions phase then there would be r_{t_1} individuals in $C(I)$. We further consider two possibilities.

(1) Suppose that there are at least $r_{t_2} + 1$ individuals in $C(I)$. Then at the last Step K, $\Delta(K) = -1$.

Individual i cannot be chosen at Case a, since this would imply that another individual should have been chosen if both remaining reservations are zero since i' is available and has higher priority than i or that if only one remaining reservation is positive, then the last individual chosen with this trait has lower priority than i'. Likewise, i cannot be chosen at Case b.b, since i' is available and has higher priority than i. Therefore, when i is chosen at a step, say k', $\Delta(k') \geq 0$. Note that $k' \neq K$ because $\Delta(K) = -1$. Hence, there is a last step, say k^* such that $\Delta(k^*) = 0$ and $\Delta(k^*+1) = -1$.

At Step k^*, $r_{t_1}(k^*), r_{r_2}(k^*) > 0$ and an individual with both traits must be chosen so that $\Delta(k^*+1) = -1$. By construction, $\Delta(k) < 0$ for every $k > k^*$. Let $\bar{i}$ be the last individual with both traits who is chosen from I. This individual must be chosen at Step k^* or after. Therefore, by construction, the priority ranking of $\bar{i}$ is weakly higher than an individual with at least one trait who is chosen when the remaining reservations for both traits are zero and that individual must have a strictly higher priority than i'. Furthermore, by construction, i must have a weakly higher priority than $\bar{i}$ since $\bar{i}$ is the last individual with both traits who is chosen. We get a contradiction that i must have a strictly higher priority than i'.

(2) Suppose that there are at most r_{t_2} individuals with trait t_2 in $C(I)$. Then i' must also have t_2 because $n\left((C(I) \setminus \{i\}) \cup \{i'\}\right) \geq n(C(I))$. Therefore, $\tau(i') = \{t_2\}$. Furthermore, there must be at least $r_{t_1} + 1$ individuals in $C(I)$ who have trait t_1.

In the last Step K, suppose for contradiction that $r_{t_2}(K) = 1$. Then the last individual to be chosen, say ℓ, has trait t_2. Furthermore, since there are at least $r_{t_1} + 1$ individuals in $C(I)$ with trait t_1, $r_{t_1}(K) = 0$. This implies that ℓ is chosen by Case a and, therefore, this individual has a higher priority than i' who also has trait t_2. If ℓ also has trait t_1, then we get a contradiction because i has a weakly higher priority than ℓ. Suppose that ℓ does not have trait t_1. Consider the last individual i^* with trait t_1 to be chosen. At the step in which i^* is chosen, the remaining reservations for t_1 is zero. Furthermore, there is a positive number of reservations for t_2 and there is at least one remaining individual with trait t_2. Therefore, i^* also has t_2. But this implies that i has a weakly higher priority than i^*, who in turn has a strictly higher priority than ℓ since ℓ is available but not chosen. This is a contradiction since ℓ has a strictly higher priority than i'.

Hence, $r_{t_2}(K) \neq 1$, which implies that $r_{t_2}(K) = 0$ and $\Delta(K) = -1$. If $\Delta(k) < 0$ for every Step k, then we get a contradiction because i must be chosen by Case a for one of the traits which cannot be t_2 because i' also has trait t_2 and i' has a higher priority than i. If i is chosen by case a because the remaining reservations for t_1 is positive then an individual with trait t_1 is chosen later when there are no remaining reservations for t_1. Hence this individual has a higher priority than i', which is a contradiction. Therefore, it cannot be that $\Delta(k) < 0$ for every Step k.

Let k^* be the last step such that $\Delta(k^*) = 0$, so $\Delta(k) < 0$ for every $k > k^*$ and $\Delta(k^* + 1) = -1$. At Step k^*, $r_{t_1}(k^*) > 0$, $r_{t_2}(k^*) > 0$, and an individual with both traits must have been chosen so that $\Delta(k^* + 1) = -1$. Let i^* be the last individual with both traits who is chosen. Therefore, i^* is chosen at a Step k where $k \geq k^*$ such that $\Delta(k) \leq 0$. Furthermore, i^* must be chosen by Case a or Case b.b because if she was chosen by one of the subcases of Case b.c, then at the next step, the value of $\Delta(\cdot)$ would be 1. If i^* is chosen by Case a because of trait t_1 then the last individual chosen with trait t_1 has a weakly higher priority than i^*, who in turn has a weakly lower priority than i, which is a contradiction. Likewise, if i^* is chosen by Case a because of trait t_2 or when no reservations remain, we get a similar contradiction. However, if i^* is chosen by Case b.b because he has the highest priority, we get a contradiction because i^* has a weakly lower priority than i.

Case 2: Consider the case when i does not have trait t_2, so $\tau(i) = \{t_1\}$. We further consider two cases.

(1) Suppose that there are at least $r_{t_2} + 1$ individuals in $C(I)$ who have trait t_2. Since $n((C(I) \setminus \{i\}) \cup \{i'\}) \geq n(C(I))$, there are also at least $r_{t_1} + 1$ individuals in $C(I)$ who have trait t_1. Furthermore, when i is chosen, there must be positive reservations for trait t_1 because i' is available and not chosen at this step. If the last chosen individual with trait t_1, say $\bar{i}$ who has to be different from i since there are at least $r_{t_1} + 1$ individuals with trait t_1 in $C(I)$, does not have trait t_2, then we get a contradiction as this individual must have strictly lower priority than i' and there are no remaining reservations for t_1. If $\bar{i}$ has trait t_2 but there are no remaining reservations for trait t_2, we get a similar contradiction. Therefore, $\bar{i}$ must have both traits and when she is chosen there are remaining reservations for trait t_2. Then the last chosen individual with trait t_2, say i^* who has to be different than $\bar{i}$, does not have t_1 and there are no remaining reservations for either trait. We get a contradiction because i^* has a weakly lower priority than $\bar{i}$ and $\bar{i}$ has a weakly lower priority than i. But i^* is chosen while i' is rejected when there are no remaining reservations, so i^* must have a strictly higher priority than i'.

(2) Consider the case when there are at most r_{t_2} individuals in $C(I)$ with trait t_2. If there are at most r_{t_1} individuals in $C(I)$ who have trait t_1, then i' must have trait t_2 and the number of individuals in $C(I)$ with trait t_2 must be strictly less than r_{t_2} because $n((C(I) \setminus \{i\}) \cup \{i'\}) \geq n(C(I))$. But at the step when i is chosen, we must have either Case b.c.b or Case b.c.d.b.b.a. If the procedure reaches either case then the chosen set of individuals $C(I)$ will have at least r_{t_2} individuals with trait t_2, which is a contradiction.

    However, if there are at least $r_{t_1} + 1$ individuals in $C(I)$ who have trait t_1, then when the lowest priority individual in $C(I)$ who has trait t_1 is chosen if i' has trait t_2, then i' should have been chosen because i' has a strictly higher priority than i, which is a contradiction. But if i' does not have any trait, then the individual in $C(I)$ with the $r_{t_1} + 1$-th highest priority who has trait t_1 must also have trait t_2. Call this individual i^*. But when i is chosen at an earlier Step k', we must have $r_{t_1}(k'), r_{t_2}(k') > 0$. If $\Delta(k') > 0$, then because of Case b.a, the highest priority remaining individual who has both traits should have been chosen, which is a contradiction since i^* has both traits and is available at that step. If $\Delta(k') < 0$, then because of Case b.b, the highest priority remaining individual should have been chosen, which is also a contradiction since i' is available. If $\Delta(k') = 0$, then i must have been chosen by Case b.c.d.b.b.a, which implies that i has a strictly higher priority than i^*. Furthermore, at this step there are more reservations for trait t_1 than trait t_2. Therefore, there must be a last step k^* such that $r_{t_1}(k^*) = r_{t_2}(k^*) > 0$ and at which a trait-t_1 individual is chosen because at the step when i^* is chosen the remaining reservations for trait t_2 is positive whereas no reservations remain for trait t_1. At Step k^* we must be at Case b.c.d.b.b.c.a. Furthermore, at this step an individual who has both traits must be chosen, which is a contradiction.

■

Lemma 10. *Every choice rule that maximally complies with reservations, eliminates justified envy, and is non-wasteful has to be a paired-admissions choice rule.*

Proof. Let C be a choice rule that maximally complies with reservations, eliminates justified envy, and is non-wasteful. Let I be a set of individuals. If $|I| \leq q$, then any non-wasteful choice rule selects I. Therefore, we assume that $|I| \geq q$ and $|C(I)| = q$.

We prove the claim by mathematical induction on q. For the base case, when $q = 1$, we consider three possibilities. Since $q = 1$, there is only one paired-admissions choice rule. Let i be the chosen individual by this choice rule.

(1) If $r_{t_1}, r_{t_2} > 0$, then $\Delta(1) > 0$. Therefore, i has the highest priority among individuals who maximize $\delta(\cdot|\emptyset)$. If $C(I)$ maximally complies with reservations, then the individual in $C(I)$ and i have the same number of traits. Furthermore, if $C(I) \neq \{i\}$, then i has a strictly higher priority than the individual in $C(I)$, so there is an instance of justified envy. Therefore, $C(I) = \{i\}$.
(2) If only reservations for one of the traits, say t_1 without loss of generality, is positive, then individual i chosen by the paired-admission choice rule Case a has the highest priority among individuals who maximize $\delta(\cdot|\emptyset)$. If $C(I)$ maximally complies with reservations, then either the individual in $C(I)$ and i both have t_1 or they both do not have t_1. Furthermore, if $C(I) \neq \{i\}$, then i has a strictly higher priority then the individual in $C(I)$, so there is an instance of justified envy. Therefore, $C(I) = \{i\}$.
(3) If $r_{t_1}, r_{t_2} = 0$, then the individual with the highest priority i must be chosen so that there is no instance of justified envy. Therefore, $C(I) = \{i\}$.

Suppose that the hypothesis is true up to capacity q. Now we show it for q. We consider two possibilities depending on the case at the first step of the paired-admission choice correspondence.

(1) Suppose that at the first step of the paired-admission choice correspondence only one individual i is chosen in the individual-admissions phase. Then $i \in C(I)$ otherwise either $C(I)$ does not maximally comply with the reservations for I or there is an instance of justified envy. Likewise i is in any paired-admission choice rule. Now consider the reduced market when $I \setminus \{i\}$ is the set of individuals, $q-1$ is the capacity, and positive reservations for a trait is reduced by one if i has that trait. In this reduced reservation market, $C(I) \setminus \{i\}$ is non-wasteful, maximally complies with reservations for $I \setminus \{i\}$, and eliminates justified envy. Therefore, by mathematical induction hypothesis, $C(I) \setminus \{i\}$ is equal to the outcome of a paired-admission choice rule for the reduced reservation market. Hence, $C(I)$ is equal to the outcome of a paired-admission choice rule for the original reservation market.

(2) Suppose that at the first step of the paired-admissions choice correspondence, all individuals are chosen at the paired-admissions phase. Thus, $r_{t_1} = r_{t_2} = r > 0$, $|I_{t_1}|, |I_{t_2}| > r$, and $q = 2r$. Furthermore, the highest priority individual does not have any traits and there exists an individual with both traits. Finally, no individual among the r highest priority individuals who have trait t_1 and r highest priority individuals who have trait t_2 has both traits. Let i be the highest priority individual with no traits and i' be the highest priority individual with both traits. If $C(I)$ has an individual with no traits, then $i \in C(I)$ to eliminate justified envy. To maximally comply with reservations, an individual with both traits must also be in $C(I)$. Therefore, $i' \in C(I)$ to eliminate justified envy. Hence, if $C(I)$ has an individual with no traits, then $\{i, i'\} \subseteq C(I)$. Now, consider the reduced reservation problem with the set of individuals $I \setminus \{i, i'\}$, capacity $q-2$, reservations for both traits $r-1$. Then in the reduced problem, $C(I) \setminus \{i, i'\}$ maximally complies with the reservations, eliminates justified envy, and is non-wasteful. By the mathematical induction hypothesis, $C(I) \setminus \{i, i'\}$ is the outcome of a paired-admission choice rule. Since $\{i, i'\}$ is selected by one of the paired-admission choice rules in the original reservation problem, we conclude that $C(I)$ is a paired-admission choice rule. The second possibility is that $C(I)$ does not have any individuals with no traits. In this case, $C(I)$ also cannot have any individual with both traits because there are r individuals with trait t_1 only and r individuals with trait t_2 only who have a higher priority than the highest priority individual with both traits. Therefore, $C(I)$ must have all these individuals, which is the outcome of a paired-admission choice rule.

■

This completes the proof of Theorem 2. □

Proof of Proposition 2. Let $I \subseteq \mathcal{I}$ be a set of applicants and C a paired-admission choice rule. We prove each statement separately.

Proof of (1a): Let $i \in C(I)$ with $\tau(i) = \emptyset$ or $\tau(i) = \{t_1, t_2\}$. If i is chosen in the individual-admissions phase, $i \in C^{\min\max}(I)$ because the same set of individuals is chosen for every paired-admission choice rule in the individual-admissions phase. However, if i is chosen in the paired-admissions phase, then $i \in C^{\min\max}(I)$ because $C^{\min\max}$ chooses the maximum number of pairs including an individual with no traits and an individual with both traits.

Proof of (1b): Let $i \in C^{\min\max}(I)$ with $\tau(i) = \{t_1\}$ or $\tau(i) = \{t_2\}$. If i is chosen in the individual-admissions phase, $i \in C(I)$ because the same set of individuals is chosen for every paired-admission choice rule in the individual-admissions phase. However, if i is chosen in the paired-admissions phase, then $i \in C(I)$ because $C^{\min\max}$ chooses the minimum number of pairs including an individual with only trait t_1 and an individual with only trait t_2.

Proof of (1c): If the paired-admission choice correspondence does not have a paired-admissions phase, then the claim is trivial. Likewise if $|I| \leq q$, the claim is trivial because there is only one paired-admissions choice rule. Suppose that the correspondence ends at the paired-admissions phase and $|I| > q$. Then the highest priority individual in $I \setminus C^{\min\max}(I)$ is either i_{m^*+1} or j_{m^*+1}. If $C(I) \neq C^{\min\max}(I)$, the highest priority individual in $I \setminus C^{\min\max}(I)$ has a weakly higher priority than i'_{m^*}. By construction of the correspondence, i'_{m^*} has a strictly higher priority than both i_{m^*+1} and j_{m^*+1}.

Proof of (2a): Let $i \in C(I)$ with $\tau(i) = \{t_1\}$ or $\tau(i) = \{t_2\}$. If i is chosen in the individual-admissions phase, $i \in C^{\max\min}(I)$ because the same set of individuals is chosen for every paired-admission choice rule in the individual-admissions phase. However, if i is chosen in the paired-admissions phase, then $i \in C^{\max\min}(I)$ because $C^{\max\min}$ chooses the maximum number of pairs including an individual with only trait t_1 and an individual with only trait t_2.

Proof of (2b): Let $i \in C^{\max\min}(I)$ with $\tau(i) = \emptyset$ or $\tau(i) = \{t_1, t_2\}$. If i is chosen in the individual-admissions phase, $i \in C(I)$ because the same set of individuals is chosen for every paired-admission choice rule in the individual-admissions phase. However, if i is chosen in the paired-admissions phase, then $i \in C(I)$ because $C^{\max\min}$ chooses the minimum number of pairs including an individual with no traits and an individual with both traits.

Proof of (2c): If the paired-admission choice correspondence does not have a paired-admissions phase or if $|I| \leq q$, then the claim is trivial because there is one paired-admissions choice rule. Suppose that the correspondence ends at the paired-admissions phase and $|I| > q$. Then the lowest priority individual in $C^{\max\min}(I)$ is either i_r or j_r. If $C(I) \neq C^{\min\max}(I)$, then the lowest priority individual in $C(I)$ has a weakly lower priority than j_1'. By construction of the correspondence, both i_r and j_r have a strictly higher priority than j_1'. □

Proof of Proposition 3. If either i or i' is chosen in the individual-admissions phase, then the other one is also chosen in the individual admissions phase, which implies that they are chosen under all paired-admissions choice rules. However, if i and i' are both chosen in the paired-admissions phse, then i and i' belong to the same pair q or p described in the construction of the paired-admission choice correspondence. By construction, for any paired-admission choice rule, the intersection of the outcome of this choice rule when I is the set of applicants and the pair is either the empty set or the pair. The conclusion follows. □

Proof of Proposition 4. The irrelevance of rejected individuals condition is satisfied trivially. We show that $C^{\boxtimes}(I)$ satisfies the substitutes condition.

Let $I \subseteq \mathcal{I}$, $i \in C^{\boxtimes}(I)$, and $i' \in I$ such that $i' \neq i$. To prove the substitutes condition, we show that $i \in C^{\boxtimes}(I \setminus \{i'\})$. If $i' \notin C^{\boxtimes}(I)$ or $i \; \pi \; i'$, then $i \in C^{\boxtimes}(I \setminus \{i'\})$ is satisfied trivially by the construction of $C^{\boxtimes}$.

Now suppose that $i' \; \pi \; i$ and $i' \in C^{\boxtimes}(I)$. If i is chosen at Step 2 of $C^{\boxtimes}(I)$, then, when the set of individuals is $I \setminus \{i'\}$, it will be chosen either at the first step or the second step. If i' is not matched at Step 1 when I is the set of applicants, then $i \in C^{\boxtimes}(I \setminus \{i'\})$. Let i' be matched with a reserved position for trait t at Step 1. The first steps of $C^{\boxtimes}(I)$ and $C^{\boxtimes}(I \setminus \{i'\})$ are the same until the priority ranking of the individual considered is below that of i'. If an individual with a priority higher than i is chosen from $I \setminus C^{\boxtimes}(I)$ when $I \setminus \{i'\}$ is the set of applicants, then the first such individual can be matched with a reserved position for trait t that i' was matched with because this individual is not chosen by $C^{\boxtimes}(I)$. After this individual is chosen, the construction of $C^{\boxtimes}(I \setminus \{i'\})$ proceeds as in the construction of $C^{\boxtimes}(I)$. Therefore, i is chosen in this case. However, if no individual with a priority higher than i is chosen from $I \setminus C^{\boxtimes}(I)$ in the construction of $C^{\boxtimes}(I \setminus \{i'\})$, then i will still be chosen. □

Proof of Proposition 5. Consider the following reservation market:

- $\mathcal{I} = \{i_1, i_2, i_3, i_4\}$,
- $\mathcal{T} = \{t_1, t_2\}$,
- $i_1 \; \pi \; i_2 \; \pi \; i_3 \; \pi \; i_4$,
- $\tau(i_1) = \emptyset$, $\tau(i_2) = \{t_1\}$, $\tau(i_3) = \{t_1, t_2\}$, $\tau(i_4) = \{t_2\}$,
- $r_{t_1} = r_{t_2} = 1$, and $q = 2$.

When the set of applicants is $\mathcal{I}$, the paired-admissions choice correspondence works as follows. At Step 1, we are at Case b.c.d.b.b.c.a, so individual i_3 is chosen. At Step 2, we are at Case a and individual i_1 is chosen. Therefore, the algorithm terminates at the individual-admissions phase and chooses $\{i_1, i_3\}$. Therefore, all paired-admission choice rules have to select this set. Likewise, when the set of applicants is $\{i_1, i_2, i_4\}$, the paired-admissions choice correspondence works as follows. At Step 1, we are at Case b.c.a, so the highest priority individual who has at least one trait, individual i_2, is chosen. At Step 2, we are at Case a, and individual i_4 is chosen. Therefore, the algorithm terminates at the individual-admissions phase and chooses $\{i_2, i_4\}$, which has to be the outcome for all paired-admission choice rules. We conclude that no paired-admission choice rule C can satisfy the substitutes condition because

$$i_1 \in C(\mathcal{I}) = \{i_1, i_3\} \text{ and } i_1 \notin C(\mathcal{I} \setminus \{i_3\}) = \{i_2, i_4\}.$$

$\square$

Proof of Proposition 6. Let $I \subseteq \mathcal{I}$ be a set of applicants and C a paired-admission choice rule. We prove each statement separately.

Proof of (1a): Let $i \in (C^{hor} \circ C)(I)$ with $\tau(i) = \{p, \ell, m\}$. If $i \in C(I)$, then by Proposition 2, $i \in C^{\min\max}(I)$, which implies $i \in (C^{hor} \circ C^{\min\max})(I)$. Otherwise, if $i \notin C(I)$, then i must have been chosen at the second step of $C^{hor} \circ C$. If $i \in C^{\min\max}(I)$, we are done. Suppose that $i \notin C^{\min\max}(I)$. This implies that every individual in $C^{\min\max}(I)$ has a higher priority than i because $\tau(i) = \{p, \ell, m\}$. Since i must have been chosen at the second step of $C^{hor} \circ C$, every individual with a higher priority ranking than i must also be in $(C^{hor} \circ C)(I)$. Therefore, $C^{\min\max}(I) \subseteq (C^{hor} \circ C)(I)$. Since individuals with the highest priority ranking is chosen at the second step of both $C^{hor} \circ C$ and $C^{hor} \circ C^{\min\max}$, this implies $(C^{hor} \circ C)(I) = (C^{hor} \circ C^{\min\max})(I)$, so $i \in (C^{hor} \circ C^{\min\max})(I)$.

Proof of (1b): Let $i \in (C^{hor} \circ C^{\min\max})(I)$ with $\tau(i) = \{p, \ell\}$ or $\tau(i) = \{p, m\}$. Without loss of generality assume that $\tau(i) = \{p, \ell\}$. If $i \in C^{\min\max}(I)$, then $i \in C(I)$ by Proposition 2, which implies that $(C^{hor} \circ C)(I)$. Suppose that $i \notin C^{\min\max}(I)$, then i must have been chosen at the second step of $C^{hor} \circ C^{\min\max}$ and all individuals with a higher priority than i must also be in $(C^{hor} \circ C^{\min\max})(I)$. If, for every $i' \in C(I)$, $i' \pi i$, then $i \in (C^{hor} \circ C)(I)$ because there are no individuals in $C(I)$ who are ranked lower than i whereas there is at least one individual with the set of traits $\{p, \ell, m\}$ in $C^{\min\max}(I)$ who is ranked lower than i. If there is an individual $i' \in C(I)$ with $\tau(i') = \tau(i)$ such that $i \pi i'$, then $i \in C(I)$, which implies $(C^{hor} \circ C)(I)$. Suppose that for every $i' \in C(I)$ with $\tau(i') = \tau(i)$, $i' \pi i$. Therefore, there exists a number of individuals in $C(I)$ with the set of traits $\{p, m\}$ who have lower priority than i. Let n be this number. By construction, there are at least n pairs in $C^{\min\max}(I)$ which have an individual with the set of traits $\{p, \ell, m\}$. These individuals have a lower priority than every individual in $C(I)$. Therefore, there are at least n individuals in $C^{\min\max}(I)$ who have a lower priority than i. Hence, $i \in (C^{hor} \circ C^{\min\max})(I)$ implies that $i \in (C^{hor} \circ C)(I)$ because the number of positions that are filled in the second step are the same for both choice rules and the number of individuals with a priority lower than i that are admitted at the first step are weakly higher in $(C^{hor} \circ C^{\min\max})(I)$ than that in $(C^{hor} \circ C)(I)$.

Proof of (2a): Let $i \in (C^{hor} \circ C)(I)$ with $\tau(i) = \{p, \ell\}$ or $\tau(i) = \{p, m\}$. If $i \in C(I)$, then $i \in C^{\max\min}(I)$ by Proposition 2, which implies that $(C^{hor} \circ C^{\max\min})(I)$. Likewise if $i \in C^{\max\min}(I)$, then we get $(C^{hor} \circ C^{\max\min})(I)$. Suppose that $i \notin C^{\max\min}(I)$ and $i \notin C(I)$. Then i must have been chosen at the second step of $C^{hor} \circ C$ and all individuals with a higher priority than i must also be in $(C^{hor} \circ C)(I)$. Since $i \notin C^{\max\min}(I)$, this implies $C^{\max\min}(I) \subseteq (C^{hor} \circ C)(I)$. Therefore, $i \in (C^{hor} \circ C^{\max\min})(I)$ because there must be enough capacity at the second step of $C^{hor} \circ C^{\max\min}$ because $C^{hor} \circ C^{\max\min}(I)$ does not have any pairs including low-priority individuals with the set of traits $\{p, \ell, m\}$ whereas $C(I)$ may have some.

Proof of (2b): Let $i \in (C^{hor} \circ C^{\max\min})(I)$ with $\tau(i) = \{p, \ell, m\}$. If $i \in C^{\max\min}(I)$, then $i \in C(I)$ by Proposition 2, which implies $i \in (C^{hor} \circ C)(I)$. Suppose that $i \notin C^{\max\min}(I)$, then i must have been chosen at the second step of $(C^{hor} \circ C^{\max\min})(I)$, and all individuals with a higher priority than i must also have been chosen. Since $C(I)$ may have some individuals with the set of traits $\tau(i) = \{p, \ell, m\}$ who have a lower priority than i, $i \in (C^{hor} \circ C)(I)$ because the number of individuals in $(C^{hor} \circ C)(I)$ and $(C^{hor} \circ C^{\max\min})(I)$ are the same.

Proof of (2c): Let i be the lowest priority individual in $(C^{hor} \circ C^{\max\min})(I)$. If $i \in C^{\max\min}(I)$, then i has a weakly higher priority than the lowest priority individual in $C(I)$ by Proposition 2, which implies that i has a weakly higher priority than the lowest priority individual in $(C^{hor} \circ C)(I)$. If $i \notin C^{\max\min}(I)$ and $C(I) \subseteq (C^{hor} \circ C^{\max\min})(I)$, then $(C^{hor} \circ C^{\max\min})(I) = (C^{hor} \circ C)(I)$ and the claim follows. If $i \notin C^{\max\min}(I)$ and $C(I) \nsubseteq (C^{hor} \circ C^{\max\min})(I)$, then there is an individual with the set of traits $\{p, \ell, m\}$ in $C(I)$ who has a lower priority than i, which implies that i has a higher priority than the lowest priority individual in $(C^{hor} \circ C)(I)$. $\square$