Leverage effects and stochastic volatility in spot oil returns: A Bayesian approach with VaR and CVaR applications[†]

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Abstract

Crude oil markets have been quite volatile and risky in the past few decades due to the large fluctuations of oil prices. We contribute to the current debate by testing for the existence of the leverage effect when considering daily spot returns in the WTI and Brent crude oil markets and by studying the direct impact of the leverage effect on measures of risk such as VaR and CVaR. More specifically, we model spot crude oil returns using Stochastic Volatility (SV) models with various distributions of the errors. We find that the introduction of the leverage effect in the traditional SV model with Normally distributed errors is capable of adequately estimating risk for conservative oil suppliers in both the WTI and Brent markets while it tends to overestimate risk for more speculative oil suppliers. Our results also show that the choice of financial regulators, both on the supply and on the demand side, would not be affected by the introduction of leverage. Focusing instead on firm's internal risk management, our results show that the introduction of leverage would be useful for firms who are on the demand side for oil, who use VaR for risk management and who are particularly worried about the magnitude of the losses exceeding VaR while wanting to minimize the opportunity cost of capital. Using the same logic, firms who are on the supply side, would be better off not considering the leverage effect.

Keywords: Value-at-Risk, Conditional Value-at-Risk, Asymmetric Laplace distribution, Stochastic volatility model, Bayesian Markov Chain Monte Carlo, leverage effect
JEL Classifications: C11, C58, G17, G32

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1. Introduction

Crude oil markets have been quite volatile and risky in the past few decades due to the large fluctuations of oil prices. This has become a principal concern for oil suppliers, oil consumers, relevant firms and governments. In addition, as a primary source of energy in the power industry, industrial production and transportation, volatile oil prices may lead to cost uncertainties for other markets, thus extensively affecting the development of the economy. A large number of studies have shown that oil price fluctuations could have considerable impact on economic activities. Papapetrou (2001) argues that the variability of oil prices plays a critical role in affecting real economic activity and employment. Lardic and Mignon (2008) explore the long-term relationship between oil prices and GDP, and find evidence that aggregate economic activity seems to slow down particularly when oil prices increase. This asymmetry is found in both the U.S. and European countries. Consequently, quantifying and managing the risks inherent to the volatility of oil prices has become critical for both researchers and energy market participants.

The Value at Risk (VaR) measure, which was first proposed by J.P. Morgan in the RiskMetrics model in 1994, has been developed as one of the most popular approaches in financial markets to manage market risk. VaR defines the maximum amount that an investor can face for a given tolerance level over a certain time horizon. Although VaR is recommended by Basel II and III and has been widely adopted by financial institutions, it has been challenged by the Bank of International Settlements (BIS) Committee, who pointed out that VaR cannot measure market risk as it fails to consider the extreme tail events of a return distribution (see, Chen et al., 2012). In addition, Artzner et al. (1999) argue that VaR does not meet the requirements of sub-additivity and thus is not a coherent measure of risk. As an alternative, they proposed a conservative, but more coherent measure, called Conditional VaR at risk (CVaR) or expected shortfall (ES), which considers the average loss as that exceeding the VaR threshold. Given all these factors, in this paper, both measures are used to quantify financial risks affecting oil markets. Existing literature that uses VaR and CVaR to measure these risks generally focuses on the scenario of a declining oil price (i.e. downside risk). However, the oil market has its own traits which are quite different from those of financial assets. More specifically, when oil prices fall due to sudden negative news, countries exporting oil or oil producers would inevitably incur losses while oil consumers would benefit from those negative extreme events. On the other hand, if oil prices rise suddenly, oil consumers would suffer a financial loss. Therefore, in this paper we consider risks affecting both oil supply and oil demand.

In recent years, the commodity price literature has shown that there is evidence of leverage effects in various energy markets. More specifically, Chan and Grant (2016a), considering lower frequency (weekly) commodity returns conclude that SV models (with an MA component) are able to replicate the main features of the data more efficiently than GARCH models. At the same time, they find a significant negative leverage effect in crude oil spot markets. Kristoufek (2014) focuses on the leverage effect in commodity futures markets and provides an extensive literature review in this area. Fan et al. (2008) estimate VaR of crude oil prices using a GED-GARCH approach with daily WTI and Brent prices spanning from 1987 to 2006. They find that this type of model specification does as well as the standard normal distribution at a 95% confidence level. They also test and find evidence for asymmetric leverage effects without modelling them directly. Youssef et al. (2015) evaluate VaR and CVar for crude oil and gasoline markets using a long memory GARCH-EVT approach. Their findings and backtesting exercise show that crude oil markets are characterized by asymmetry, fat tails and long range memory.

We contribute to the current debate by testing for the existence of the leverage effect when considering daily spot returns in the WTI and Brent crude oil markets and by studying the direct impact of the leverage effect on measures of risk such as VaR and CVaR. More specifically, in order to address the risk faced by oil suppliers and oil consumers we model spot crude oil returns using Stochastic Volatility (SV) models with various distributions of the errors. Among other cases, we test the assumption of Asymmetric Laplace Distributed (ALD) errors in order to more carefully model the type of risk faced by oil suppliers versus the risk faced by oil buyers.

We find that the introduction of the leverage effect in the traditional SV model with Normally distributed errors is capable of adequately estimating risk (in a VaR and CVaR sense) for conservative (i.e. more risk averse, with $\alpha = 5\%$) oil suppliers in both the WTI and Brent markets while it tends to overestimate risk for more speculative oil suppliers ($\alpha = 1\%$). In comparison, the assumption of ALD errors leads to overestimating risk for both types of investors. In the model efficiency selection stage, our results show that the choice of financial regulators, both on the supply and on the demand side, would not be affected by the introduction of leverage. Focusing instead on firm's internal risk management, our results show that the introduction of leverage (SV-N-L model) would be useful for firms who are on the demand side for oil (in both the WTI and Brent markets), who use VaR for risk management and who are particularly worried about the magnitude of the losses exceeding VaR while wanting to minimize the opportunity cost of capital. Using the same logic, firms who are on the supply side, would be better off not considering the leverage effect (SV-N model).

2. Stochastic volatility models

We use a general SV model to capture the volatility features for oil markets which has been studied recently by Takahashi et al. (2009), Chai et al. (2011) and Chan et al. (2016a):

$$y_t = \mu + \sigma_t z_t \tag{1}$$

$$ln\sigma_t^2 = h_t = \delta + \beta(ln\sigma_{t-1}^2 - \delta) + \eta_t \qquad \eta_t \sim N(0, \sigma_\eta^2) \tag{2}$$

where y_t denotes stock returns at time t with $t = 1, 2, ..., T$, μ denotes the conditional mean, σ_t is the stochastic volatility, $ln\sigma_t^2$ follows a stationary AR(1) process with persistence parameter β having $|\beta| < 1$, z_t and η_t represent a series of independent identical (*i.i.d.*) random errors in the return and volatility equation, respectively.[2]

For this general equation, we consider various possible specifications of the shocks z_t affecting stock returns.

(1) Standard Student t errors

$$z_t \sim t_\nu$$

where ν is the degrees of freedom of t-distribution.

(2) Standard Normal errors

$$z_t \sim N(0, 1)$$

(3) Standard Asymmetric Laplace errors

$$z_t \sim ALD(0, \kappa, 1)$$

where $\sigma = 1$ and κ is the coefficient driving the skewness of the distribution, is related to μ and σ as follows:

$$\mu = \frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa} - \kappa\right)$$

as a special case $\kappa = 1$ for $\mu \simeq 0$ and $\sigma = e^{\frac{h_t}{2}} > 0$ (*Symmetric Laplace Distribution*).[3]

[2] A number of original empirical works via extended SV models can be found from Breidt et al. (1998), So et al. (1998), Yu and Yang (2002), Koopman and Uspensky (2002), Cappuccio et al. (2004), Chan (2013), Chan and Hsiao (2013), Chan and Grant (2016c), Chan (2017).

[3] See appendix for the density of ALD.

(4) Standard Student t errors with leverage effect

$$
\begin{aligned}
y_t &= \mu + \sigma_t z_t \\
z_t &\sim t_\nu \\
\ln \sigma_t^2 = h_t &= \delta + \beta \left(\ln \sigma_{t-1}^2 - \delta \right) + \xi_t \\
\xi_t &= \rho z_t + \sqrt{1-\rho^2} \eta_t \\
\eta_t &\sim N \left(0, \sigma_\eta^2 \right)
\end{aligned}
$$

where the coefficient ρ drives the so called *leverage effect*. It models the correlation between the shocks affecting returns and the shocks affecting volatility. For example, a negative ρ would mean that negative shocks to returns are likely to be associated to positive shocks to volatility: negative shocks to financial markets would trigger higher volatility and riskiness. Of course for $\rho = 0$, the model would be simply the regular SV-t model with no leverage effect.

(5) Standard Normal errors with leverage effect

$$
\begin{aligned}
y_t &= \mu + \sigma_t z_t \\
z_t &\sim N \left(0, 1 \right) \\
\ln \sigma_t^2 = h_t &= \delta + \beta \left(\ln \sigma_{t-1}^2 - \delta \right) + \xi_t \\
\xi_t &= \rho z_t + \sqrt{1-\rho^2} \eta_t \\
\eta_t &\sim N \left(0, \sigma_\eta^2 \right)
\end{aligned}
$$

where ρ is the coefficient driving the leverage effect in the SV-N-L model.

(6) Standard Asymmetric Laplace distributed errors with leverage effect

$$
\begin{aligned}
y_t &= \mu + \sigma_t z_t \\
z_t &\sim ALD \left(0, \kappa, 1 \right) \\
\ln \sigma_t^2 = h_t &= \delta + \beta \left(\ln \sigma_{t-1}^2 - \delta \right) + \xi_t \\
\xi_t &= \rho z_t + \sqrt{1-\rho^2} \eta_t \\
\eta_t &\sim N \left(0, \sigma_\eta^2 \right)
\end{aligned}
$$

where ρ is the coefficient driving the leverage effect in the SV-ALD-L model.

3. VaR and CVaR models

Considering $VaR_{s,t}(l)$ and $VaR_{d,t}(l)$ as the VaR for oil supply and demand in l-period with confidence level $(1-\alpha) \in (0,1)$ respectively, then, we have:

$$Supply: \quad Prob\,(y_t(l) \leq -VaR_{s,t}(l)|\Omega_t) = \alpha \tag{3}$$

$$Demand: \quad Prob\,(y_t(l) \geq VaR_{d,t}(l)|\Omega_t) = \alpha \tag{4}$$

where $y_t(l)$ represents the oil return series for period (from t to $t+l$), Ω_t is the information set up to time t, α is the risk level, and the value of $VaR_{s,t}$ and $VaR_{d,t}$ are defined to be positive. Likewise, $CVaR_{s,t}(l)$ and $CVaR_{d,t}(l)$ are defined as the CVaR of oil supply and demand respectively over period l at confidence level $(1-\alpha)$, and they can be mathematically expressed as:

$$Supply: \quad CVaR_{s,t}(l) = -E\{y_t(l)|y_t(l) \leq -VaR_{s,t}(l)\} \tag{5}$$

$$Demand: \quad CVaR_{d,t}(l) = E\{y_t(l)|y_t(l) \geq VaR_{d,t}(l)\} \tag{6}$$

3.1. In the SV-N setting

Now we introduce the VaR and CVaR formulas under the SV-N framework.

Risk for oil Supply

(1) VaR: $\quad VaR_{n,s,t} = -\mu - \sigma_t \Phi^{-1}(\alpha)$

where $\Phi^{-1}(\alpha)$ is the inverse cumulative distribution function of a N(0,1). In order to model the leverage effect in this setting, we use $\sigma_t(\rho)$.

(2) CVaR: $\quad CVaR_{n,s,t} = -E\left[y_t|\, y_t \leq -VaR_{_{n,s,t}}\right] = -\mu - \dfrac{\sigma_t}{\alpha}\phi(\Phi^{-1}(\alpha))$

where $\phi(\alpha)$ is the probability density function of a N (0,1). To model the leverage effect in this setting, we use $\sigma_t(\rho)$.

Risk for oil demand

(1) VaR: $\quad VaR_{n,d,t} = \mu + \sigma_t \Phi^{-1}(\alpha)$

where $\Phi^{-1}(\alpha)$ is the inverse cumulative distribution function of a N(0,1). To model the leverage effect in this setting, we use $\sigma_t(\rho)$.

(2) CVaR: $\quad CVaR_{n,d,t} = E\left[y_t|\, y_t \geq VaR_{_{n,d,t}}\right] = \mu + \dfrac{\sigma_t}{\alpha}\phi(\Phi^{-1}(\alpha))$

where $\phi(\alpha)$ is the probability density function of a N (0,1). To model the leverage effect in this setting, we use $\sigma_t(\rho)$.

3.2. In the SV-ALD setting

We now introduce the VaR and CVaR formulas under the SV-ALD model.

Risk for oil Supply

(1) VaR: $$VaR_{s,t} = -\mu + m_{s,q}\sigma_t = -\mu - \frac{\kappa\sigma_t}{\sqrt{2}} ln\frac{\alpha(1+\kappa^2)}{\kappa^2}$$

where $m_{s,q} = (VaR_{s,t} + \mu)/\sigma_t$ is defined as the left α-quantile of the AL distribution. In order to model the leverage effect in this setting, we use $\sigma_t(\rho)$.

(2) CVaR: $$CVaR_{s,t} = -E\left[y_t | \, y_t \leq -VaR_{s,t}\right] = VaR_{s,t} + \frac{\kappa\sigma_t}{\sqrt{2}}$$

To model the leverage effect in this setting, we use $\sigma_t(\rho)$.

Risk for oil Demand

(1) VaR: $$VaR_{d,t} = \mu + m_{d,q}\sigma_t = \mu - \frac{\sigma_t}{\sqrt{2}\kappa} ln(\alpha(1+\kappa^2))$$

where $m_{d,q} = (VaR_{d,t} - \mu)/\sigma_t$ is the right α-quantile of the AL distribution. To model the leverage effect in this setting, we use $\sigma_t(\rho)$.

(2) CVaR: $$CVaR_{d,t} = E\left[y_t | \, y_t \geq VaR_{d,t}\right] = VaR_{d,t} + \frac{\sigma_t}{\sqrt{2}\kappa}$$

To model the leverage effect in this setting, we use $\sigma_t(\rho)$.[4]

4. Estimation methodology of Bayesian MCMC

In order to improve the tractability of the ALD model, we introduce a modified Bayesian MCMC method. That is, a new Scale Mixture of Uniform (SMU) representation for the AL density (following Kotz et al., 2001) is proposed to facilitate the estimation of the SV-ALD model.

[4]See appendix B for the derivations of $VaR_{s,t}$, $VaR_{d,t}$, $CVaR_{s,t}$ and $CVaR_{d,t}$.

4.1. Scale mixture of uniform representation of ALD

Expressing the ALD via the representation can alleviate the computational burden when using the Gibbs sampling algorithm in the MCMC approach and thus can simplify the estimation method in Bayesian analysis. To estimate the latent variables in the SV model, we use the scaled ALD (SALD) which means that the ALD random variable is scaled by its standard deviation (See Chen et al., 2009 and Wichitaksorn et al., 2015).

Proposition 1. *Let z_t be the ALD random variable with $z_t \sim ALD(0, \kappa, 1)$, then the random variable $\varepsilon_t = \frac{z_t}{S.D.[z]}$ has SALD with p.d.f. given by:*

$$f(\varepsilon_t|\kappa, \sigma_t) = \begin{cases} \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t}\ exp(\frac{-\sqrt{1+\kappa^4}}{\sigma_t}\varepsilon_t) & \varepsilon_t \geq 0 \\ \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t}\ exp(\frac{\sqrt{1+\kappa^4}}{\kappa^2\sigma_t}\varepsilon_t) & \varepsilon_t < 0 \end{cases} \tag{7}$$

where κ is skewness parameter and σ_t is the standard deviation (or the time-varying volatility) of z_t.[5]

Hence, the corresponding SMU of SALD can be obtained as follows:

Proposition 2. *If $\lambda_t \sim Ga(2, 1)$ and $\varepsilon_t \sim U(\varepsilon_t| - \frac{\lambda_t\kappa^2\sigma_t}{\sqrt{1+\kappa^4}}, +\frac{\lambda_t\sigma_t}{\sqrt{1+\kappa^4}})$, then the SMU density:*

$$f(\varepsilon_t|\kappa, \lambda_t, \sigma_t) = \int_0^\infty f_U(\varepsilon_t| - \frac{\lambda_t\kappa^2\sigma_t}{\sqrt{1+\kappa^4}}, +\frac{\lambda_t\sigma_t}{\sqrt{1+\kappa^4}}) \times f_{Ga}(\lambda_t|2, 1)\, d\lambda_t \tag{8}$$

has the same form as the SALD density function given in equation (7).[6]

Using the SMU representation of SALD, an efficient simulation algorithm is developed to overcome parameter estimation difficulties. As a result, the SV model discussed in section 2 can be written hierarchically as:
Return equation:

$$y_t|\kappa, \lambda_t, h_t \sim U(- \frac{\lambda_t\kappa^2 e^{h_t/2}}{\sqrt{1+\kappa^4}}, +\frac{\lambda_t e^{h_t/2}}{\sqrt{1+\kappa^4}}) \tag{9}$$

$$\lambda_t \sim Ga(2, 1) \tag{10}$$

[5]Note that original scale parameter has been canceled in this derivation, while the location parameter θ is set to be 0 in real practice. See appendix C for the derivation.

[6]See appendix D for the derivation.

Volatility equation:

$$h_t|\delta,\beta,\sigma_\eta^2,h_{t-1} \sim N(\delta+\beta(h_{t-1}-\delta),\sigma_\eta^2) \qquad t=1,2,...,T \tag{11}$$

$$h_1|\delta,\beta,\sigma_\eta^2 \sim N(\delta,\frac{\sigma_\eta^2}{1-\beta^2}) \tag{12}$$

4.2. Bayesian Markov Chain Monte Carlo

We employ the Bayesian MCMC approach via the Gibbs sampling algorithm to make posterior inference of SV-ALD model as an SMU. To implement MCMC, we set priors as:

$$\delta \sim N(\mu_\delta,\frac{1}{\sigma_\delta^2}); \qquad \frac{1+\beta}{2}=\beta^* \sim Be(a_\beta,b_\beta); \qquad \sigma_\eta^2 \sim IG(a_\sigma,b_\sigma)$$

where $Be(\cdot,\cdot)$ denotes beta distribution and $IG(\cdot,\cdot)$ is an inverse-gamma distribution.[7] In order to simplify the Gibbs sampling algorithm, the full conditional distribution of parameters via the SMU of ALD must be derived. Thus, the Gibbs sampler mimics a random sample from the intractable joint posterior distribution by iteratively simulating random variables from the system of full conditional distributions.[8]

5. VaR and CVaR evaluation methods

5.1. Accuracy measures

In order to backtest the accuracy and appropriateness of the estimated VaRs, we calculate the empirical failure rates for both oil supply and demand. The *failure rate (FR)* is the ratio of the number of times that oil returns exceed the estimated VaRs over the number of observations. The model is said to be correctly specified if the calculated ratio is equal to the pre-specified VaR level α (i.e. 5% and 1%). If the ratio is greater than α, we conclude that the model underestimate the risks, and vice versa. In addition, three likelihood ratio backtesting criteria are implemented to test the statistical accuracy of the methods. These criteria include unconditional coverage test (LR_{uc} by Kupiec, 1995), independent test (LR_{ind} by Christoffersen, 1998) and conditional coverage test (LR_{cc} by Christoffersen, 1998).

For backtesting CVaR in the SV model with Normal errors, we select the nominal risk level at 5% and 1% as 1.96% and 0.38%, following the work by Chen et al. (2012). To identify the nominal risk level for ALD, we use a cumulative distribution function ($c.d.f.$) of

[7]Note that for the prior setting in the Bayesian inference, the WinBugs software uses a nonstandard parametrization of Normal distribution in terms of the precision ($1/variance$) instead of the variance.

[8]See appendix E for the derivation of the parameters of full conditional posterior distributions.

ALD which, according to Kotz et al. (2001), is given by:

$$F(z|\kappa,\theta,\sigma) = \begin{cases} 1 - \dfrac{1}{1+\kappa^2}\ exp\Big(-\dfrac{\sqrt{2}\kappa}{\sigma} z\Big) & z \geq 0 \\ \dfrac{\kappa^2}{1+\kappa^2}\ exp\Big(\dfrac{\sqrt{2}}{\sigma\kappa} z\Big) & z < 0 \end{cases} \tag{13}$$

Then, this *c.d.f.* is evaluated at the point that equates to the CVaR level. As a consequence, the probability ($\tilde{\alpha}$) that CVaR occurs under ALD for oil supply and demand can be mathematically expressed as:

$$Supply: \quad \tilde{\alpha} = F(CVaR_s|\alpha) = \frac{\alpha}{e} \tag{14}$$

$$Demand: \quad \tilde{\alpha} = 1 - F(CVaR_d|\alpha) = \frac{\alpha}{e} \tag{15}$$

where e is the natural exponent. For both of oil supply and demand, the quantile level of CVaR under ALD is simply a function of α and e and does not depend on other parameters in the AL density. This surprising finding is consistent with results of Chen et al. (2012) although different ALD forms have been studied. Hence, according to formula (14) and (15), the nominal risk level $\tilde{\alpha}$ for CVaR under ALD at 5% and 1% are obtained as 1.84% and 0.37%, respectively. As a consequence, using $\tilde{\alpha}$ as prescribed risk level for CVaR backtesting, the statistics test LR_{uc}, LR_{ind} and LR_{cc} can be run to examine accuracy of CVaR model.

5.2. Efficiency measures

Adhering to the Basel Committee's guidelines, supervisors are not only concerned with the quantities of violations in a VaR model but also with the magnitude of those violations (Basel Committee on Banking Supervision, 1996a, 1996b). Hence, we employ the regulatory loss function (RLF) and firm's loss function (FLF) of Sarma et al. (2003), which considers both the number of violations and their magnitude. This is a two-stage model evaluation procedure where the first stage aims to test the models in terms of statistical accuracy, while in the second stage the models surviving the statistical accuracy tests are then evaluated for efficiency (details see Sarma et al., 2003).

6. Simulation experiment

To examine the effectiveness of the proposed MCMC sampling procedure, we estimate the SV-ALD model using simulated data. We generate 2874 observations from the SV-ALD model given by (1) and (2) with ALD errors by fixing parameter values $\delta = -7.587$, $\beta = 0.9947$, $\sigma_\eta = 0.0889$ and $\kappa = 0.9956$. The true parameter values are chosen as the parameter estimates from the empirical study of WTI market.

Table 1: MCMC estimation results for the SV-ALD model for the simulated data

Parameter	True	Mean	Median	SD	MC errors	95% CI
δ	-7.58700	-7.92400	-7.92300	0.90820	0.00647	(-9.48900, -6.35700)
β	0.99470	0.99600	0.99610	0.00190	0.00006	(0.99160, 0.99910)
σ_η	0.08890	0.12880	0.125700	0.01610	0.00091	(0.10600, 0.17430)
κ	0.99560	0.97630	0.975100	0.01190	0.00068	(0.95610, 1.00200)

The following prior distribution are assumed: $\delta \sim N(-10, 0.001)$ with $0.001 = 1/\sigma_\delta^2$; $\tau_\eta \sim Ga(2.5, 0, 025)$ with $\tau_\eta = 1/\sigma_\eta^2$; $\beta^* \sim Be(20, 1.5)$ with $\beta^* = (\beta + 1)/2$, and the prior distribution of skewness parameter was chosen as: $\kappa \sim U(0, 2)$, where τ_η is precision parameter, $Ga(\cdot, \cdot)$, $Be(\cdot, \cdot)$ and $U(\cdot, \cdot)$ represents gamma distribution, beta distribution and uniform distribution, respectively. In the Gibbs sampling scheme, 3 Markov chains are run from diverse starting positions for 40000 iterations. The initial 30000 iterations are discarded as the burn-in period to ensure convergence and the remaining 3×10000=30000 samples are used for estimation and inference.

Table 1 shows the posterior means, posterior standard deviations, Monte Carlo errors and 95% credible intervals for the parameters. The posterior means and medians for β, σ_η and κ are very close to the true values with small MC errors, which are all located inside the 95% credible intervals. The posterior means and medians for δ vary more than others as it would be expected since the variance parameter is quantile-dependent (similar findings also see Chen et al., 2009).

7. Empirical analysis

7.1. Data

We consider two major crude oil markets: West Texas Intermediate crude oil (WTI) and Europe Brent crude oil (Brent). Daily closing spot prices, which are quoted in US dollars per barrel, are obtained from the U.S. Energy Information Administration (EIA) covering the periods from May 22, 2006 to May 20, 2016, resulting in 2520 observations in WTI and 2522 observations in Brent. p_t denotes the oil price at day t, $y_t = ln(p_t/p_{t-1})$ is the daily return. The time-variations of daily prices and returns for WTI and Brent are given in Figure 1. The graphs of daily returns show the existence of volatility clustering in both the two oil markets which revealing the presence of heteroscedasticity. This phenomenon is particularly evident during the global financial crisis. The largest changes of oil returns for WTI and Brent occurred on September 22, 2008 and January 2, 2009 with a record of 16.41% and 18.13% surge respectively.

Descriptive statistics for WTI and Brent returns are provided in Table 2. From Panel A, the Jarque--Bera test indicates the non-Gaussian features of return series. Both WTI

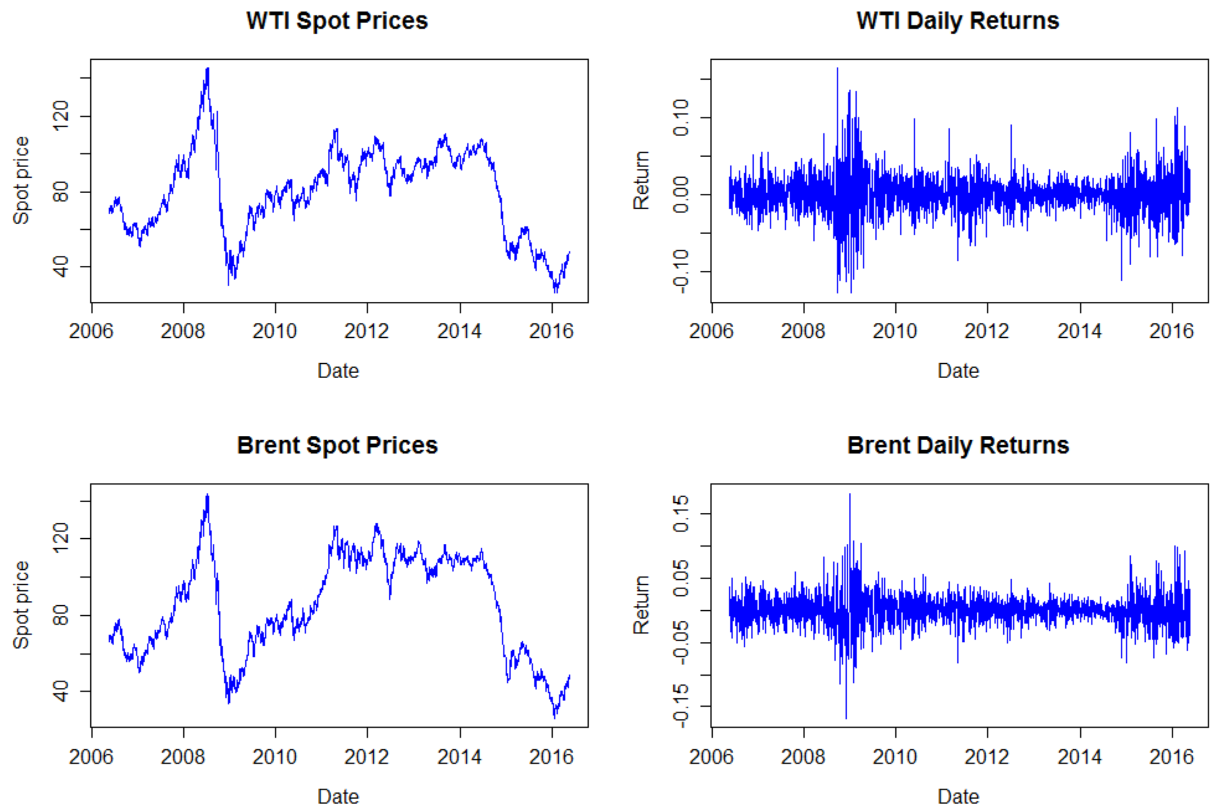

Figure 1: Daily spot prices and returns for WIT and Brent from May 1987 to May 2016

and Brent markets show a mild positive skewness and are leptokurtic or "fat-tailed" with a significant kurtosis greater than 3. Results from the Ljung--Box Q statistics of order up to 20 indicate the existence of autocorrelation in the datasets. The daily returns series exhibit significant ARCH effects at 10 and 20 lags at 1% significance level. This aspect will be taken into account when examining the estimation results. These results can also be immediately observed from the pattern of return series in Figure 1 where large price movements are followed by large movements.

Moreover, three tests are employed to examine the stationarity of the time series before fitting them. Augmented Dicky--Fuller (ADF) test and Phillips--Perron (PP) test significantly reject the null hypothesis of unit root, and the statistics from the Kwiatkowski--Phillips--Schmidt--Shin (KPSS) test show that we cannot reject the assumption of stationarity of the series.

7.2. SV models estimation and comparisons

Convergence diagnostic. Before estimating the parameters from the joint posterior distribution, convergence diagnostics of the constructed Markov chains in the MCMC algorithm are conducted using the Brooks--Gelman--Rubin (BGR) diagnostic approach. To

Table 2: Descriptive statistics for WTI and Brent oil price returns

	WTI	Brent
Panel A: Descriptive statistics		
Mean	-0.000144	-0.000127
Std.dev.	0.024863	0.021998
Maximum	0.164137	0.181297
Minimum	-0.128267	-0.168320
Skewness	0.1567	0.1443
Kurtosis	7.6122	8.8043
J-B test	2243.0570***	3547.5790***
Q(10)	30.6030***	16.9600*
Q(20)	60.8980***	54.2270***
ARCH(10)	475.9680***	215.7230***
ARCH(20)	575.8620***	409.0370***
Panel B: Unit roots and stationarity tests		
ADF	-51.4930***	-48.9570***
PP	-51.5220***	-48.9660***
KPSS	0.0507	0.0690

Note: Q(l) are Ljung--Box statistics for up to lth order serial correlation. Test statistics of ARCH are obtained using chi-squared distribution. ADF and PP are statistics of the Augmented Dickey--Fuller and Phillips--Perron unit root tests. The largest value from the first 8 lags of KPSS test is listed. $*$, $**$ and $***$ denote rejection of null hypothesis at 10%, 5% and 1% significant level respectively.

complete a Bayesian paradigm, the prior distribution of the estimated SV model parameters are set as: $\delta \sim N(-10, 0.001)$ with $0.001 = 1/\sigma_\delta^2$; $\tau_\eta \sim Ga(2.5, 0, 025)$ with $\tau_\eta = 1/\sigma_\eta^2$; $\beta^* \sim Be(20, 1.5)$ with $\beta^* = (\beta + 1)/2$, and the prior distribution of skewness parameter are chosen as: $\kappa \sim U(0, 2)$, where τ_η is precision parameter, $Ga(\cdot, \cdot)$, $Be(\cdot, \cdot)$ and $U(\cdot, \cdot)$ represents gamma distribution, beta distribution and uniform distribution, respectively.[9] After discarding the corresponding burn-in period in each SV-type models, the remaining simulated samples are used to construct posterior inferences.

[9] The sensitivity analysis indicates that the changing of prior numbers have minor influence on the posterior distributions of the parameters (δ, β and σ_η). The setting priors here are those which can facilitate our posterior inference and enable the speed of convergence to become faster.

Table 3: Posterior summary statistics for the parameters in SV-t, SV-N and SV-ALD models

Market	Parameter	Mean	SD	MC error	95% CI
		SV-t			
WTI	ν	13.42000	2.83100	0.13450	(9.24600,20.10000)
	δ	-9.77200	0.34470	0.00298	(-10.44000,-9.08500)
	β	0.99830	0.00101	0.00002	(0.99580,0.99970)
	σ_η	0.09595	0.01143	0.00060	(0.07470,0.11840)
Brent	ν	12.61000	2.50600	0.11710	(8.78900,18.43000)
	δ	-9.76700	0.34190	0.00300	(-10.43000,-9.09200)
	β	0.99850	0.00094	0.00002	(0.99620,0.99980)
	σ_η	0.08333	0.01086	0.00058	(0.06549,0.10730)
		SV-N			
WTI	μ	0.00037	0.00034	0.00000	(-0.00030,0.00103)
	δ	-9.76100	0.34660	0.00298	(-10.43000,-9.06800)
	β	0.99780	0.00122	0.00003	(0.99490,0.99950)
	σ_η	0.11760	0.01360	0.00072	(0.09209,0.14630)
Brent	μ	0.00013	0.00031	0.00000	(-0.00046,0.00074)
	δ	-9.78700	0.34210	0.00277	(-10.45000,-9.10000)
	β	0.99860	0.00089	0.00002	(0.99640,0.99980)
	σ_η	0.08933	0.01019	0.00054	(0.07112,0.11030)
		SV-ALD			
WTI	κ	0.99560	0.01592	0.00083	(0.96590,1.02800)
	δ	-7.58700	0.48730	0.00408	(-8.42200,-6.69800)
	β	0.99470	0.00224	0.00005	(0.98990,0.99870)
	σ_η	0.08891	0.00992	0.00051	(0.07065,0.10810)
Brent	κ	0.99820	0.01206	0.00061	(0.97510,1.02200)
	δ	-7.75000	0.56200	0.00587	(-8.66000,-6.69600)
	β	0.99590	0.00184	0.00004	(0.99200,0.99910)
	σ_η	0.07351	0.00812	0.00042	(0.06106,0.09410)

Table 4: Posterior summary statistics for the parameters in SV-t-L, SV-N-L and SV-ALD-L models

Market	Parameter	Mean	SD	MC error	95% CI
			SV-t-L		
WTI	ρ	-0.62110	0.07616	0.00422	(-0.75940, -0.47680)
	ν	10.90000	1.63300	0.06735	(8.22400, 14.50000)
	δ	-9.71700	0.35290	0.00376	(-10.39000, -9.01300)
	β	0.99830	0.00091	0.00002	(0.99610, 0.99960)
	σ_η	0.09112	0.01027	0.00058	(0.07171, 0.10950)
Brent	ρ	-0.57170	0.07303	0.00396	(-0.69400, -0.42250)
	ν	12.38000	2.79900	0.13860	(8.56100, 19.59000)
	δ	-9.75300	0.34420	0.00327	(-10.42000, -9.07100)
	β	0.9986	0.00084	0.00002	(0.99650, 0.99970)
	σ_η	0.08100	0.00946	0.00053	(0.06535, 0.10170)
			SV-N-L		
WTI	ρ	-0.54850	0.07225	0.00387	(-0.66870,-0.39170)
	μ	-0.00009	0.00035	0.00001	(-0.00078,0.00059)
	δ	-9.73200	0.35170	0.00310	(-10.41000,-9.02700)
	β	0.99810	0.00100	0.00002	(0.99570,0.99960)
	σ_η	0.11110	0.01034	0.00057	(0.09117,0.13000)
Brent	ρ	-0.62630	0.05649	0.00300	(-0.74130,-0.51490)
	μ	-0.00025	0.00031	0.00000	(-0.00085,0.00035)
	δ	-9.78400	0.34170	0.00304	(-10.45000,-9.10800)
	β	0.99880	0.00070	0.00001	(0.99710,0.99980)
	σ_η	0.08544	0.00821	0.00045	(0.07289,0.10570)
			SV-ALD-L		
WTI	ρ	-0.74780	0.05345	0.00303	(-0.83640,-0.63140)
	κ	1.00100	0.01336	0.00077	(0.97690,1.02700)
	δ	-7.75400	0.38370	0.00485	(-8.46500,-7.12300)
	β	0.99550	0.00156	0.00004	(0.99230,0.99840)
	σ_η	0.09288	0.00826	0.00047	(0.07945,0.10980)
Brent	ρ	-0.67460	0.06573	0.00369	(-0.78440,-0.53440)
	κ	1.00700	0.01282	0.00073	(0.98060,1.02900)
	δ	-7.91800	0.53680	0.00447	(-8.93000,-6.97100)
	β	0.99690	0.00151	0.00005	(0.99340,0.99930)
	σ_η	0.07427	0.00942	0.00055	(0.06148,0.09575)

Table 5: WTI: In sample Root Mean Square Error (RMSE)

Year	RMSE SV-t	RMSE SV-t-L	RMSE SV-N	RMSE SV-N-L	RMSE SV-ALD	RMSE SV-ALD-L
2006	0.024666	0.024832	0.024440	0.024392	0.026936	0.026238
2007	0.026485	0.026972	0.026365	0.026705	0.028254	0.029127
2008	0.054977	0.055920	0.054435	0.055954	0.058000	0.056521
2009	0.048405	0.048945	0.048270	0.048405	0.050172	0.049486
2010	0.026399	0.025732	0.026114	0.025974	0.027611	0.027094
2011	0.030133	0.030007	0.030416	0.030359	0.031569	0.031547
2012	0.022471	0.022367	0.022619	0.022762	0.023780	0.024324
2013	0.016566	0.016461	0.016389	0.016542	0.017860	0.017806
2014	0.023295	0.022629	0.023508	0.023643	0.024341	0.024142
2015	0.041619	0.042051	0.041430	0.041656	0.044052	0.044278
2016	0.054839	0.054196	0.054817	0.055393	0.057046	0.057881

Table 6: WTI: In sample Mean Absolute Error (MAE)

Year	MAE SV-t	MAE SV-t-L	MAE SV-N	MAE SV-N-L	MAE SV-ALD	MAE SV-ALD-L
2006	0.019466	0.019450	0.019307	0.019271	0.020534	0.020089
2007	0.020686	0.021100	0.020693	0.021011	0.021565	0.022337
2008	0.038980	0.039382	0.038741	0.039792	0.040603	0.039382
2009	0.035243	0.035479	0.035168	0.035312	0.035673	0.035263
2010	0.020427	0.019900	0.020269	0.020146	0.020922	0.020547
2011	0.023174	0.023026	0.023541	0.023491	0.023819	0.023741
2012	0.017130	0.017094	0.017255	0.017409	0.017748	0.018131
2013	0.013029	0.012923	0.012926	0.013086	0.013686	0.013678
2014	0.016440	0.015911	0.016646	0.016715	0.016746	0.016608
2015	0.032297	0.032592	0.032250	0.032387	0.033461	0.033497
2016	0.042475	0.041849	0.042657	0.042804	0.043476	0.043773

Table 7: Brent: In sample Root Mean Square Error (RMSE)

Year	RMSE SV-t	RMSE SV-t-L	RMSE SV-N	RMSE SV-N-L	RMSE SV-ALD	RMSE SV-ALD-L
2006	0.028946	0.028129	0.028704	0.028340	0.030493	0.028612
2007	0.025080	0.025013	0.024659	0.024954	0.026579	0.028343
2008	0.043941	0.044021	0.044793	0.045493	0.045327	0.051346
2009	0.044852	0.045047	0.044703	0.045426	0.046725	0.048226
2010	0.025725	0.026107	0.025390	0.025443	0.026979	0.026727
2011	0.024890	0.025037	0.024733	0.024698	0.025505	0.028801
2012	0.020495	0.020322	0.020519	0.020438	0.021459	0.023273
2013	0.015774	0.015605	0.015715	0.015712	0.016467	0.017586
2014	0.016807	0.017026	0.016870	0.017287	0.017466	0.020727
2015	0.036134	0.035715	0.035869	0.036038	0.036561	0.041760
2016	0.048917	0.048831	0.048579	0.048513	0.048577	0.054604

Table 8: Brent: In sample Mean Absolute Error (MAE)

Year	MAE SV-t	MAE SV-t-L	MAE SV-N	MAE SV-N-L	MAE SV-ALD	MAE SV-ALD-L
2006	0.022680	0.022025	0.022675	0.022358	0.023294	0.021928
2007	0.019715	0.019656	0.019550	0.019745	0.020408	0.021612
2008	0.031998	0.031628	0.032659	0.032980	0.032348	0.035790
2009	0.033279	0.033293	0.033103	0.033729	0.034127	0.034435
2010	0.019849	0.020170	0.019725	0.019820	0.020345	0.020362
2011	0.019328	0.019429	0.019319	0.019302	0.019335	0.021650
2012	0.015935	0.015718	0.016062	0.015953	0.016367	0.017591
2013	0.012273	0.012124	0.012324	0.012326	0.012578	0.013379
2014	0.012167	0.012278	0.012205	0.012469	0.012485	0.014310
2015	0.028058	0.027767	0.028071	0.028192	0.027736	0.031315
2016	0.038214	0.038177	0.038241	0.038077	0.037361	0.041129

Posterior estimates and model comparisons. To compare the fitting ability of SV-ALD and SV-ALD-L model with conventional SV-N, SV-N-L, SV-t and SV-t-L models, results of posterior estimates of these models are shown in Table 3 and 4.[10] The posterior means of β in WTI and Brent markets under these models are very close to one, which is consistent with our general beliefs that there exist a strong persistence of volatility in oil returns. Our results show that the estimated posterior mean of σ_η for the SV-ALD model is lower comparing to the corresponding σ_η in the SV-N and SV-t model in both of the two oil markets, and σ_η in the SV-t model is lower than that σ_η in the SV-N model.

The estimate for the σ_η parameter from the SV-ALD-L and SV-t-L model is lower than the estimate coming from the SV-N-L model. These results are consistent with the findings from Chib et al. (2002) and Abanto-Valle et al. (2010), indicating that the introduction of heavy tailed error distribution in the mean equation appears to explain excess returns, thus decreasing the variance of the volatility process. More importantly, we find statistically significant negative correlation ($\rho < 0$) between shocks affecting oil returns and shocks affecting volatility in the SV-t-L, SV-N-L and the SV-ALD-L specifications. Although WinBugs can generate deviance information criterion (DIC) values straightforwardly, as pointed out by Chan and Grant (2016), conditional DIC typically favors over-fitted models in a series of Monte Carlo experiments. Therefore this cannot be used as a reliable criterion to compare across models. For this reason, we use various comparison criteria. First of all we check the in sample RMSE and MAE calculated by year in Table 5 to Table 8.[11] Considering the in sample RMSE and MAE, the SV-N and SV-N-L models outperform the others. Because this result is not conclusive, in the next sections, we also consider additional criteria such as

[10] We used the WinBUGS's code from the website of Yasuhiro Omori as the starting point.

[11] $\widehat{y_t}$ is replicated by using the Bayesian estimates for the model parameters and for the volatility.

Table 9: Out-of-sample performance for various models: RMSE and MAE for May-December 2016

Market	SV-t	SV-t-L	SV-N	SV-N-L	SV-ALD	SV-ALD-L
			RMSE			
WTI	0.034910	0.034952	0.033276	0.033010	0.038417	0.035692
Brent	0.038329	0.037781	0.038010	0.037333	0.037907	0.038116
			MAE			
WTI	0.026940	0.026973	0.025831	0.025362	0.028911	0.026794
Brent	0.029482	0.029095	0.029476	0.028731	0.028495	0.028391

the out-of-sample RMSE and MAE (to test the predictive power of the models) and, more importantly, the capability of the models to replicate risk in a VaR and CVaR sense.

Out-of-sample performance. Table 9 shows the out-of-sample performance for various models using the Root Mean square errors (RMSE) and Mean Absolute errors (MAE) criteria. We used the MCMC estimates from May 2006 to May 2016 to forecast oil returns from the end of May 2016 to the end of December 2016: the SV-N-L model performs better than its competitors for both markets if we consider the RMSE criterion (calculated using 500 simulations and fixing the parameters at the MCMC estimates). Considering the MAE criterion, SV-ALD-L and SV-N-L perform the best.

Table 10 to Table 15 present the results of Engle's LM ARCH test on the standard errors for SV-t, SV-t-L, SV-N, SV-N-L, SV-ALD and SV-ALD-L models in both markets. Considering the series of standard errors, there is no evidence of ARCH effects for the SV-N model while the SV-N-L model shows ARCH effects in the WTI market at a 1% significance level. This result gives an opportunity to increase efficiency by modeling ARCH, but does not violate any assumptions made when estimating the underlying model. As a conclusion, the SV-N model is the most efficient among the set of models that have been studied in this paper. From Table 16 to Table 21, we can see that the Kolmogorov Smirnov test for normality does not reject its null for the Brent standard errors resulting from the SV-t, SV-t-L, SV-N and SV-N-L models at the 1% significance level. For the WTI standard errors, it does not reject its null hypothesis of normality for the SV-N and SV-N-L standard errors at 1% significance level. The Shapiro Francia test (1972) for normality concurs with those judgements for the standard errors coming from all the models. The Box Pierce portmanteau (or Q) test for white noise rejects its null for both series of standard errors.

Diebold Mariano test. This test calculates a measure of predictive accuracy proposed by Diebold and Mariano (1995). We ran the test for each of 500 simulations per model

Table 10: WTI: Engle's Lagrange multiplier test for autoregressive conditional heteroskedasticity for standardised residuals and squared standardised residuals for SV-t and SV-t-L models

	1 lag	p-val	5 lags	p-val	10 lags	p-val	30 lags	p-val
SV-t res	2.04	0.15	8.98	0.11	13.75	0.18	47.15	0.02
SV-t res squ	0.19	0.66	0.82	0.98	1.66	1.00	11.50	1.00
SV-t-L res	3.51	0.06	9.46	0.09	12.79	0.24	39.48	0.12
SV-t-L res squ	0.09	0.76	0.57	0.99	0.98	1.00	31.60	0.39

Table 11: WTI: Engle's Lagrange multiplier test for autoregressive conditional heteroskedasticity for standardised residuals and squared standardised residuals for SV-N and SV-N-L models

	1 lag	p-val	5 lags	p-val	10 lags	p-val	30 lags	p-val
SV-N res	0.00	0.95	16.21	0.01	27.56	0.00	77.20	0.00
SV-N res squ	0.03	0.87	1.99	0.85	4.29	0.93	19.90	0.92
SV-N-L res	0.24	0.63	11.73	0.04	20.67	0.02	60.18	0.00
SV-N-L res squ	0.07	0.79	1.23	0.94	3.07	0.98	18.77	0.94

Table 12: WTI: Engle's Lagrange multiplier test for autoregressive conditional heteroskedasticity for standardised residuals and squared standardised residuals for SV-ALD and SV-ALD-L models

	1 lag	p-val	5 lags	p-val	10 lags	p-val	30 lags	p-val
SV-ALD res	7.31	0.01	9.72	0.08	11.20	0.34	35.24	0.23
SV-ALD res squ	0.63	0.43	1.04	0.96	1.49	1.00	13.45	1.00
SV-ALD-L res	13.69	0.00	15.75	0.01	18.05	0.05	39.41	0.12
SV-ALD-L res squ	7.55	0.01	8.13	0.15	8.62	0.57	36.25	0.20

Table 13: Brent: Engle's Lagrange multiplier test for autoregressive conditional heteroskedasticity for standardised residuals and squared standardised residuals for SV-t and SV-t-L models

	1 lag	p-val	5 lags	p-val	10 lags	p-val	30 lags	p-val
SV-t res	5.40	0.02	17.55	0.00	26.16	0.00	50.28	0.01
SV-t res squ	0.69	0.41	3.92	0.56	5.91	0.82	33.66	0.29
SV-t-L res	5.35	0.02	14.18	0.01	21.18	0.02	43.89	0.05
SV-t-L res squ	0.51	0.47	3.36	0.64	5.18	0.88	27.70	0.59

Table 14: Brent: Engle's Lagrange multiplier test for autoregressive conditional heteroskedasticity for standardised residuals and squared standardised residuals for SV-N and SV-N-L models

	1 lag	p-val	5 lags	p-val	10 lags	p-val	30 lags	p-val
SV-N res	7.17	0.01	22.48	0.00	34.65	0.00	70.86	0.00
SV-N res squ	1.22	0.27	6.91	0.23	10.47	0.40	31.61	0.39
SV-N-L res	5.38	0.02	13.79	0.02	21.96	0.02	47.77	0.02
SV-N-L res squ	0.78	0.38	4.29	0.51	7.01	0.72	22.95	0.82

Table 15: Brent: Engle's Lagrange multiplier test for autoregressive conditional heteroskedasticity for standardised residuals and squared standardised residuals for SV-ALD and SV-ALD-L models

	1 lag	p-val	5 lags	p-val	10 lags	p-val	30 lags	p-val
SV-ALD res	0.94	0.33	6.24	0.28	9.97	0.44	26.97	0.62
SV-ALD res squ	0.10	0.75	1.42	0.92	2.64	0.99	35.76	0.22
SV-ALD-L res	2.70	0.10	8.02	0.15	11.89	0.29	28.67	0.54
SV-ALD-L res squ	0.33	0.56	1.81	0.87	2.84	0.98	14.38	0.99

Table 16: WTI: Test Statistics and P-values for standardised residuals and squared standardised residuals for SV-t and SV-t-L models

	KSmirnov	p-val	SFrancia	p-val	Qtest	p-val
SV-t res	0.011	0.940	3.514	0.000	33.011	0.775
SV-t res squ	0.263	0.000	15.592	0.000	44.168	0.300
SV-t-L res	0.019	0.346	5.281	0.000	33.544	0.755
SV-t-L res squ	0.276	0.000	15.857	0.000	43.892	0.310

Table 17: WTI: Test Statistics and P-values for standardised residuals and squared standardised residuals for SV-N and SV-N-L models

	KSmirnov	p-val	SFrancia	p-val	Qtest	p-val
SV-N res	0.009	0.988	0.414	0.340	33.470	0.758
SV-N res squ	0.245	0.000	15.165	0.000	54.062	0.068
SV-N-L res	0.016	0.576	2.627	0.004	33.652	0.750
SV-N-L res squ	0.253	0.000	15.339	0.000	47.908	0.183

Table 18: WTI: Test Statistics and P-values for standardised residuals and squared standardised residuals for SV-ALD and SV-ALD-L models

	KSmirnov	p-val	SFrancia	p-val	Qtest	p-val
SV-ALD res	0.015	0.587	4.652	0.000	32.453	0.796
SV-ALD res squ	0.273	0.000	15.766	0.000	40.367	0.454
SV-ALD-L res	0.020	0.241	5.749	0.000	34.901	0.699
SV-ALD-L res squ	0.280	0.000	15.907	0.000	45.253	0.262

Table 19: Brent: Test Statistics and P-values for standardised residuals and squared standardised residuals for SV-t and SV-t-L models

	KSmirnov	p-val	SFrancia	p-val	Qtest	p-val
SV-t res	0.022	0.177	2.716	0.003	43.504	0.325
SV-t res squ	0.257	0.000	15.268	0.000	48.892	0.158
SV-t-L res	0.024	0.115	2.820	0.002	43.906	0.310
SV-t-L res squ	0.257	0.000	15.277	0.000	46.316	0.228

Table 20: Brent: Test Statistics and P-values for standardised residuals and squared standardised residuals for SV-N and SV-N-L models

	KSmirnov	p-val	SFrancia	p-val	Qtest	p-val
SV-N res	0.021	0.201	1.865	0.031	43.025	0.343
SV-N res squ	0.250	0.000	15.102	0.000	56.556	0.043
SV-N-L res	0.023	0.156	2.340	0.010	42.681	0.357
SV-N-L res squ	0.253	0.000	15.175	0.000	47.479	0.194

Table 21: Brent: Test Statistics and P-values for standardised residuals and squared standardised residuals for SV-ALD and SV-ALD-L models

	KSmirnov	p-val	SFrancia	p-val	Qtest	p-val
SV-ALD res	0.024	0.122	4.169	0.000	43.549	0.323
SV-ALD res squ	0.266	0.000	15.449	0.000	35.039	0.693
SV-ALD-L res	0.025	0.084	3.799	0.000	43.420	0.328
SV-ALD-L res squ	0.264	0.000	15.439	0.000	35.317	0.681

Table 22: Diebold Mariano test: comparison of forecast accuracy over 500 out-of-sample predictions

Variable	Observations	Mean	SD	Min	Max
		WTI			
SV-t vs SV-t-L					
r_{1t}	41	0.0010954	0.0004075	0.0006428	0.0023901
r_{2t}	41	0.0011213	0.0003617	0.0005928	0.0022897
SV-N vs SV-N-L					
r_{1t}	241	0.0012336	0.0005985	0.0005911	0.0046736
r_{2t}	241	0.0012112	0.0008895	0.0005486	0.0103819
SV-ALD vs SV-ALD-L					
r_{1t}	240	0.001657	0.0006006	0.0007806	0.0045232
r_{2t}	240	0.0012848	0.0008002	0.0006352	0.0051321
		Brent			
SV-t vs SV-t-L					
r_{1t}	46	0.0014817	0.0004475	0.0008498	0.0028845
r_{2t}	46	0.001352	0.000371	0.000789	0.0025136
SV-N vs SV-N-L					
r_{1t}	258	0.0015684	0.0006906	0.0007161	0.0063691
r_{2t}	258	0.0014911	0.0008635	0.0006984	0.0078067
SV-ALD vs SV-ALD-L					
r_{1t}	209	0.0015392	0.0006215	0.0007717	0.0041496
r_{2t}	209	0.00158	0.000905	0.0007069	0.0090912

and present summary statistics from that set of test results. Given an actual series and two competing predictions, one may apply a loss criterion (such as squared error or absolute error) and then calculate a number of measures of predictive accuracy that allow the null hypothesis of equal accuracy to be tested. Table 22 reports the results where the r_1 and r_2 variables are the MSEs for model 1 (*non-leverage model*) and model 2 (*leverage model*), respectively. If the $p - value < 0.05$, the test rejects the null that the two models are equally capable in terms of their MSEs. For the simulations in which the test rejects equal forecast accuracy, we can compare the mean MSE for the two models.

For the WTI data, in the case of SV-t vs SV-t-L models, we can observe 41 rejections (over 500 out-of-sample simulations): model 1 (the *non-leverage model*) has the smaller mean MSE. Considering SV-N vs SV-N-L models, we can observe 241 rejections: model 2 (*the leverage model*) has the smaller mean MSE. Considering SV-ALD vs SV-N-ALD models, we can observe 240 rejections: model 2 (*the leverage model*) has the smaller mean MSE. For the Brent data, in the case of SV-t vs SV-t-L models, we can observe 46 rejections: model 2 (*the leverage model*) has the smaller mean MSE. Considering SV-N vs SV-N-L models, we can observe 258 rejections: model 2 (*the leverage model*) has the smaller mean MSE. Considering SV-ALD vs SV-N-ALD models, we can observe 209 rejections: model 1 (the *non-leverage model*) has the smaller mean MSE.

In summary, in four of the six simulations, model 2 (*the leverage model*) has the smaller mean MSE for those simulations in which the Diebold–Mariano test rejects its null hypothesis of equal forecast accuracy.

7.3. Selection of VaR and CVaR models

We now focus on the models for which we have the most evidence of a substantial impact of the introduction of leverage on the prediction accuracy of the model (SV-N, SV-N-L, SV-ALD and SV-ALD-L models). In order to classify the competing models, we follow a two-stage model evaluation procedure where in the first stage models are selected in terms of their statistical accuracy (*backtesting stage*), while in the second stage the surviving models are evaluated in terms of their efficiency (*efficiency stage*).[12]

Stage 1: Backtesting VaR and CVaR model. The *Failure Rate* (**FR**) or violation rate, computes the ratio of the number of times oil returns exceed the estimated VaRs over the total number of observations. The model is said to be correctly specified if the calculated ratio is equal to the pre-specified VaR level α (i.e. 5% and 1%). If the Failure Rate is greater than α, we can conclude that the model underestimates the risk, and vice versa (see Marimoutou et al., 2009; Aloui and Mabrouk, 2010; Louzis et al., 2014).

A criterion for evaluating our results comes from the consideration that a conservative investor (see for example, Zhao et al., 2015 and Hung et al., 2008) might choose a greater confidence level and estimate a relatively greater risk (corresponding to $\alpha = 5\%$ in the VaR definition), while a more speculative investor might estimate a smaller risk and face a relatively smaller confidence level (corresponding to $\alpha = 1\%$ in the VaR definition).

In order to backtest the accuracy of the estimated VaRs, three formal tests are conducted based on the criteria of the empirical failure rates. Table 23 shows the VaR backtesting summary results of SV-N, SV-ALD, SV-N-L and SV-ALD-L models for the WTI and Brent market, considering both supply and demand risks. According to the LR_{ind} test, the null hypothesis that exceptions are independent cannot be rejected at the two risk levels in the two markets for both the four models by considering oil supply and demand risk, suggesting that there are not many/no consecutive violations. Modelling the data using SV-N and SV-N-L models, the test LR_{uc} and LR_{cc} are passed, which indicates the capability of the models of estimating tail risks. The SV-ALD and SV-ALD-L models overestimate the tail risk and the null hypothesis of tests LR_{uc} and LR_{cc} are rejected in WTI market and partly rejected in

[12]For details see Sarma et al. (2003).

Table 23: VaR backtesting results for WTI and Brent markets

α	Risk	Failure times		Failure rate		LR_{uc}		LR_{ind}		LR_{cc}	
		WTI	Brent	WTI	Brent	WTI	Brent	WTI	Brent	WTI	Brent
					SV-N						
5%	VaR$_{st}$	107	122	4.248%	4.839%	0.0757	0.7099	0.8269	0.2114	0.1930	0.3869
	VaR$_{dt}$	111	116	4.407%	4.720%	0.1634	0.3754	0.9598	0.4968	0.3615	0.9944
1%	VaR$_{st}$	22	24	0.873%	0.952%	0.5138	0.8071	0.5334	0.5334	0.6598	0.7559
	VaR$_{dt}$	16	22	0.635%	0.873%	0.0486	0.5113	0.6510	0.6591	0.1282	0.6524
					SV-ALD						
5%	VaR$_{st}$	77	98	3.057%	3.887%	0.0000*	0.0077	0.6766	0.9211	0.0000*	0.0265
	VaR$_{dt}$	82	97	3.255%	3.848%	0.0000*	0.0057	0.4314	0.2596	0.0001*	0.0108
1%	VaR$_{st}$	7	8	0.278%	0.317%	0.0000*	0.0001*	0.8434	0.8214	0.0001*	0.0003*
	VaR$_{dt}$	5	6	0.198%	0.238%	0.0000*	0.0000*	0.8878	0.8656	0.0000*	0.0000*
					SV-N-L						
5%	VaR$_{st}$	112	126	4.446%	4.998%	0.1940	0.9964	0.3700	0.4934	0.2751	0.6781
	VaR$_{dt}$	103	109	4.086%	4.324%	0.0305	0.1111	0.3912	0.7538	0.0639	0.0105
1%	VaR$_{st}$	31	25	1.231%	0.992%	0.2615	0.9664	0.3793	0.4789	0.3572	0.7546
	VaR$_{dt}$	13	21	0.516%	0.833%	0.0071	0.3856	0.7134	0.5523	0.0249	0.5609
					SV-ALD-L						
5%	VaR$_{st}$	91	100	3.613%	3.967%	0.0008*	0.0137	0.0210	0.5969	0.0002*	0.0368
	VaR$_{dt}$	79	95	3.136%	3.768%	0.0000*	0.0031*	0.3571	0.8225	0.0000*	0.0108
1%	VaR$_{st}$	9	6	0.357%	0.238%	0.0002*	0.0000*	0.7994	0.8656	0.0009*	0.0000*
	VaR$_{dt}$	5	5	0.198%	0.198%	0.0000*	0.0000*	0.8878	0.8878	0.0000*	0.0000*

Note: $\alpha = 5\%$ and 1% represent prescribed VaR level corresponding to 95% and 99% CI respectively, LR_{uc} columns show p-values of Kupiec's (1995) unconditional coverage test, LR_{ind} columns are p-values of Christoffersen's (1998) independent test and LR_{cc} columns are p-values of Christoffersen's (1998) conditional coverage test, * denotes significance.

Brent market especially when focusing on extreme tail risks (1%). Table 24 presents CVaR backtesting results for SV-N, SV-ALD, SV-N-L and SV-ALD-L models for oil supply and demand in the WTI and Brent markets. The performance of CVaR is very similar to the VaR performance. Looking at the p-values of the SV-N and SV-N-L model, they pass the three tests for the studied risk levels.

Considering both Tables 23 and 24, the main finding is that the introduction of the leverage effect in the traditional SV model with Normally distributed errors is capable of adequately estimating risk (in a VaR and CVaR sense) for conservative (i.e. more risk averse, with $\alpha = 5\%$) oil suppliers in both the WTI and Brent markets while it tends to overestimate risk for more speculative oil suppliers ($\alpha = 5\%$). In comparison, the assumption of ALD errors

Table 24: CVaR backtesting results for WTI and Brent markets

$\tilde{\alpha}$	Risk	Failure times		Failure rate		LR_{uc}		LR_{ind}		LR_{cc}	
		WTI	Brent	WTI	Brent	WTI	Brent	WTI	Brent	WTI	Brent
					SV-N						
1.96%	CVaR_{st}	41	43	1.628%	1.706%	0.2152	0.3463	0.2440	0.2216	0.2315	0.2887
	CVaR_{dt}	33	45	1.310%	1.785%	0.0123*	0.5199	0.3492	0.2006	0.0278	0.3396
0.38%	CVaR_{st}	10	10	0.397%	0.397%	0.8906	0.8926	0.7776	0.7776	0.9481	0.9409
	CVaR_{dt}	7	9	0.278%	0.357%	0.3816	0.8496	0.8434	0.7994	0.6669	0.9408
					SV-ALD						
1.84%	CVaR_{st}	19	15	0.754%	0.595%	0.0000*	0.0000*	0.5909	0.6716	0.0000*	0.0000*
	CVaR_{dt}	11	19	0.437%	0.754%	0.0000*	0.0000*	0.7560	0.5909	0.0000*	0.0000*
0.37%	CVaR_{st}	2	2	0.079%	0.079%	0.0035*	0.0035*	0.9550	0.9550	0.0142	0.0141
	CVaR_{dt}	3	0	0.119%	0.000%	0.0155	0.0000*	0.9326	1.0000	0.0533	0.0001*
					SV-N-L						
1.96%	CVaR_{st}	45	54	1.786%	2.142%	0.5235	0.5159	0.2006	0.8780	0.3533	0.7499
	CVaR_{dt}	32	40	1.270%	1.587%	0.0077*	0.1621	0.3641	0.2558	0.0187*	0.1881
0.38%	CVaR_{st}	12	14	0.476%	0.555%	0.4495	0.1809	0.7346	0.6924	0.7060	0.3715
	CVaR_{dt}	6	9	0.238%	0.357%	0.2140	0.8496	0.8656	0.7994	0.4544	0.9408
					SV-ALD-L						
1.84%	CVaR_{st}	24	15	0.953%	0.595%	0.0003*	0.0000*	0.4967	0.6716	0.0010*	0.0000*
	CVaR_{dt}	8	17	0.318%	0.674%	0.0000*	0.0000*	0.8213	0.6307	0.0000*	0.0000*
0.37%	CVaR_{st}	3	2	0.119%	0.079%	0.0155	0.0035*	0.9326	0.9550	0.0533	0.0141
	CVaR_{dt}	3	1	0.119%	0.040%	0.0155	0.0005*	0.9326	0.9775	0.0533	0.0022*

Note: $\tilde{\alpha}$ = 1.96% and 0.38% corresponds to 5% and 1% risk level of Normal distribution and $\tilde{\alpha}$ = 1.84% and 0.37% corresponds to 5% and 1% risk level of ALD. LR_{uc} columns show p-values of Kupiec's (1995) unconditional coverage test, LR_{ind} columns are p-values of Christoffersen's (1998) independent test and LR_{cc} columns are p-values of Christoffersen's (1998) conditional coverage test, * denotes significance.

leads to overestimating risk for both type of investors.

Stage 2: Efficiency Measures. Lopez (1998, 1999) was the first to propose the comparison between VaR models on the basis of their ability to minimise some specific loss function which reflected a specific objective of the risk manager.

Adhering to the Basel Committee's guidelines, supervisors are not only concerned with the number of violations in a VaR model but also with the magnitude of those violations (Basel Committee on Banking Supervision, 1996a, 1996b). In order to address this aspect, following Sarma et al. (2003), we compare the relevant models in terms of the *Regulatory Loss Function* (**RLF**) which focuses on the magnitude of the failure and in terms of the *Firm's Loss Function* (**FLF**) which, while giving relevance to the magnitude of failures, imposes an

Table 25: RLF and FLF Loss function approach applied to the models surviving the VaR backtesting stage

Volatility models and VaR methods		RLF				FLF			
		5%		1%		5%		1%	
		WTI	Brent	WTI	Brent	WTI	Brent	WTI	Brent
Panel A: Average loss values									
SV-N	Supply	0.000209	0.000199	0.000170	0.000182	0.001709	0.001555	0.000328	0.002279
	Demand	0.000251	0.000237	0.000501	0.000227	-0.001680	-0.001531	-0.000499	-0.002274
SV-N-L	Supply	0.000239	0.000203	0.000176	0.000192	0.001755	0.001586	0.000353	0.002317
	Demand	0.000229	0.000219	0.000511	0.000162	-0.001733	-0.001569	-0.000511	-0.002313
SV-ALD	Supply	-	0.000250	-	-	-	0.001681	-	-
	Demand	-	-	-	-	-	-	-	-
SV-ALD-L	Supply	-	0.000235	-	-	-	0.001716	-	-
	Demand	-	-	-	-	-	-	-	-
Panel B: Sign statistics									
S_{AB}	Supply	47.2408	46.9433	49.1934	49.4129	-13.0107*	-11.6517*	-9.9422*	-9.2612*
	Demand	48.8746	49.0146	49.9904	49.9307	12.0144	10.6155	9.5438	8.8230
S_{BA}	Supply	48.4363	48.1382	49.9505	49.8909	13.0107	11.6517	9.9422	9.2612
	Demand	46.8822	46.5051	49.7115	49.5722	-12.0144*	-10.6155*	-9.5438*	-8.8230*
S_{CD}	Supply	-	48.1382	-	-	-	-7.1102*	-	-
	Demand	-	-	-	-	-	-	-	-
S_{DC}	Supply	-	47.8594	-	-	-	7.1102	-	-
	Demand	-	-	-	-	-	-	-	-

Note: This table compares the best performing models in the VaR backtesting procedure following the Regulatory loss function (RLF) and Firm's loss function (FLF). Panel A presents the average loss values for RLF and FLF for the competing models at different risk levels in the two oil markets. The models with the lowest average loss values are underlined. Panel B reports the standardized sign statistics values. S_{AB} denotes the standardized sign statistics with null of "non-superiority" of SV-N over SV-N-L, S_{BA} represents the standardized sign statistics with null of "non-superiority" of SV-N-L over SV-N, S_{CD} is the standardized sign statistics with null hypothesis of "non-superiority" of SV-ALD over SV-ALD-L while S_{DC} is the standardized sign statistics with null hypothesis of "non-superiority" of SV-ALD-L over SV-ALD. * means significance in the corresponding level.

additional penalty related to the opportunity cost of capital.[13] We use a non-parametric sign test to check the ability the relevant VaR models to minimize these loss functions.[14]

Table 25 presents the summary results for the RLF and FLF loss function approach as applied to the models chosen in the VaR backtesting stage. The results in Panel A show that the SV-N model achieves the smallest value of average loss more often than the SV-N-L model while the outcome is not conclusive for the SV-ALD model and the SV-ALD-L model under the two approaches. To examine the statistical significance of the losses, we report the values of the standardized sign test in Panel B. Considering the RLF criterion, this test shows that the competing models (leverage vs no-leverage models) are not significantly different from

[13]This criterion penalizes large failures more than small failures (See Sarma et al., 2003).

[14]For the sign test see Lehmann (1974), Diebold and Mariano (1995), Hollander and Wolfe (1999) and Sarma et al. (2003).

Table 26: RLF and FLF Loss function approach applied to the models surviving the CVaR backtesting stage

Volatility models and CVaR methods		RLF				FLF			
		1.84%/1.96%		0.37%/0.38%		1.84%/1.96%		0.37%/0.38%	
		WTI	Brent	WTI	Brent	WTI	Brent	WTI	Brent
Panel A: Average loss values									
SV-N	Supply	0.000167	0.000201	0.000143	0.000136	0.000877	0.000796	0.000222	0.000204
	Demand	-	-	0.000284	0.000167	-	-	-0.001221	-0.000203
SV-N-L	Supply	0.000228	0.000189	0.000226	0.000129	0.000881	0.000794	0.000224	0.000204
	Demand	-	-	0.000301	0.000100	-	-	-0.000222	-0.000203
SV-ALD	Supply	-	-	-	-	-	-	-	-
	Demand	-	-	0.000094	-	-	-	-0.000317	-
SV-ALD-L	Supply	-	-	-	-	-	-	-	-
	Demand	-	-	0.000107	-	-	-	-0.000310	-
Panel B: Sign statistics									
S_{AB}	Supply	48.5160	48.4171	49.7115	49.7714	1.3748	2.5294	0.6974	1.6531
	Demand	-	-	50.1099	50.0502	-	-	-1.0958	-1.7726
S_{BA}	Supply	49.8708	49.7714	50.1896	50.0502	-1.3748	-2.5294*	-0.6974	-1.6531
	Demand	-	-	49.9904	49.9706	-	-	1.0958	1.7726
S_{CD}	Supply	-	-	-	-	-	-	-	-
	Demand	-	-	50.1498	-	-	-	-10.8190*	-
S_{DC}	Supply	-	-	-	-	-	-	-	-
	Demand	-	-	50.0701	-	-	-	10.8190	

Note: This table compares the best performing models in the CVaR backtesting procedure following the Regulatory loss function (RLF) and Firm's loss function (FLF). Panel A presents the average loss values of RLF and FLF for the competing models at different risk levels in the two oil markets. The models with the lowest average loss values are underlined. Panel B reports the standardized sign statistics values. S_{AB} denotes the standardized sign statistics with null of "non-superiority" of SV-N over SV-N-L, S_{BA} represents the standardized sign statistics with null of "non-superiority" of SV-N-L over SV-N, S_{CD} is the standardized sign statistics with null hypothesis of "non-superiority" of SV-ALD over SV-ALD-L while S_{DC} is the standardized sign statistics with null hypothesis of "non-superiority" of SV-ALD-L over SV-ALD. The nominal risk level 1.84% and 0.37% corresponds to 5% and 1% risk level of ALD and 1.96% and 0.38% corresponds to 5% and 1% of the Normal distribution. * means significance in the corresponding nominal level.

each others which means that the choice of financial regulators, both on the supply and on the demand side, would not be affected by the introduction of leverage. Considering Panel B for the FLF criterion, for both the WTI and Brent markets, the SV-N model is significantly better than the SV-N-L model for firms involved with oil supply while the SV-N-L model is significantly better than the SV-N model for firms interested in oil demand. This means that the introduction of leverage (SV-N-L) would be useful for firms who are on the demand side for oil in both the WTI and Brent markets, who use VaR for risk management and who are particularly worried about the magnitude of the losses exceeding VaR while wanting to minimize the opportunity cost of capital. Using the same logic, firms who are on the supply side, would be better off not considering the leverage effect.

Table 26 shows the summary results of RLF and FLF loss function approach applied to the models chosen in the CVaR backtesting stage. In terms of the average economic losses and considering both RLF and FLF as selection criteria, the SV-N model performs relatively better than the SV-N-L model in the WTI market while in the Brent market, the SV-N-L model outperforms the SV-N model. The standardized sign test values by FLF in the Panel B indicate that in most cases there are no significant differences between the competitors. The only exception is that the SV-N-L model outperforms the SV-N model for oil supply in the Brent market at 1.96% risk level and the SV-ALD model performs better for oil demand in the WTI market at 0.37%

8. Conclusions

In this paper, we study the interaction between oil returns and volatility by using daily spot returns in the crude oil markets (both WTI and Brent) with a particular consideration for the impact of the leverage effect on measures of risk such as VaR and CVaR. We find that, allowing for leverage, traditional SV models with Normal distributed errors provide the best predictions in our out of sample experiments.

In order to address the risk faced by oil suppliers and oil consumers we model spot crude oil returns using Stochastic Volatility (SV) models with various error distributions. Among other cases, we test the assumption of Asymmetric Laplace Distributed (ALD) errors in order to model in a more distinctive way the type of risk faced by oil suppliers versus the risk faced by oil buyers.

We find that the introduction of the leverage effect in the traditional SV model with Normally distributed errors is capable of adequately estimating risk (in a VaR and CVaR sense) for conservative (i.e. more risk averse, with $\alpha = 5\%$) oil suppliers in both the WTI and Brent markets while it tends to overestimate risk for more speculative oil suppliers ($\alpha = 1\%$). In comparison, the assumption of ALD errors leads to overestimating risk for both type of investors. In the model efficiency selection stage, our results show that the choice of financial regulators, both on the supply and on the demand side, would not be affected by the introduction of leverage. Focusing instead on firm's internal risk management, our results show that the introduction of leverage (SV-N-L model) would be useful for firms who are on the demand side for oil (in both the WTI and Brent markets), who use VaR for risk management and who are particularly worried about the magnitude of the losses exceeding VaR while wanting to minimize the opportunity cost of capital. Using the same logic, firms who are on the supply side, would be better off not considering the leverage effect (SV-N model).

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A. Asymmetric Laplace distribution

A random variable X is said to follow an Asymmetric Laplace Distribution if the characteristic function of X can be defined as:

$$\psi(t) = \frac{1}{1 + \frac{1}{2}\sigma^2 t^2 - i\mu t} \tag{16}$$

where i is the imaginary unit, $t \in R$ is the argument of the characteristic function, σ is the scale parameter with $\sigma > 0$ and μ is the mean of X. Then, we have $X \sim AL(\mu, \sigma)$. Note that this characteristic function is a standardized form with location parameter $\theta = 0$. An equivalent notation for the distribution of X can be written as $AL(\mu, \tau)$. More details can refer to Kotz et al. (2001).

The density function is given by:

$$f(z|\kappa, \sigma, \theta) = \begin{cases} \frac{\sqrt{2}}{\tau}\frac{\kappa}{1+\kappa^2}\, exp(-\frac{\sqrt{2}\kappa}{\tau}(z-\theta)) & z \geq \theta \\ \frac{\sqrt{2}}{\tau}\frac{\kappa}{1+\kappa^2}\, exp(\frac{\sqrt{2}}{\tau\kappa}(z-\theta)) & z < \theta \end{cases} \tag{17}$$

B. VaR and CVaR derivation for oil supply and demand under SV-ALD

For oil supply, we have:

$$\begin{aligned} P(y_t \leq -VaR_{s,t}|\Omega_t) &= P\left(\frac{y_t - \mu}{\sigma_t} \leq -\frac{VaR_{s,t} - \mu}{\sigma_t}\middle|\Omega_t\right) \\ &= P\left(z_t \leq -m_{s,q} = -\frac{VaR_{s,t} + \mu}{\sigma_t}\right) = \int_{-\infty}^{-m_{s,q}} f^-(z_t)\, dz_t \\ &= \int_{-\infty}^{-m_{s,q}} \frac{\sqrt{2}}{\tau}\frac{\kappa}{1+\kappa^2}\, exp(\frac{\sqrt{2}z_t}{\tau\kappa})\, dz_t \\ &= \frac{\sqrt{2}}{\tau}\frac{\kappa}{1+\kappa^2}\int_{-\infty}^{-m_{s,q}} \frac{\tau\kappa}{\sqrt{2}} d(exp(\frac{\sqrt{2}z_t}{\tau\kappa})) \\ &= \frac{\kappa^2}{1+\kappa^2}\left(exp(\frac{\sqrt{2}(-m_{s,q})}{\tau\kappa}) - exp(\frac{\sqrt{2}(-\infty)}{\kappa\tau})\right) \\ &= \frac{\kappa^2}{1+\kappa^2} exp(\frac{\sqrt{2}(-m_{s,q})}{\kappa\tau}) = \alpha \end{aligned}$$

where $f^-(z_t)$ is the negative part of the $p.d.f.$ of ALD. Transforming this equation and setting $\tau = 1$, we can obtain the VaR for oil supply:

$$VaR_{s,t} = -\mu + m_{s,q}\sigma_t = -\mu - \frac{\kappa\sigma_t}{\sqrt{2}}\, ln\frac{\alpha(1+\kappa^2)}{\kappa^2} \tag{18}$$

and further the CVaR for oil supply:

$$CVaR_{s,t} = -\,E\left[y_t | y_t \leq -VaR_{s,t}\right] = VaR_{s,t} + \frac{\kappa\sigma_t}{\sqrt{2}} \tag{19}$$

For oil demand, we have:

$$\begin{aligned}
P(y_t > VaR_{d,t}|\Omega_t) &= P\left(\frac{y_t - \mu}{\sigma_t} > \frac{VaR_{d,t} - \mu}{\sigma_t}\middle|\Omega_t\right) \\
&= P\left(z_t > m_{d,q} = \frac{VaR_{d,t} - \mu}{\sigma_t}\right) = \int_{m_{d,q}}^{+\infty} f^+(z_t)\,dz_t \\
&= \int_{m_{d,q}}^{+\infty} \frac{\sqrt{2}}{\tau}\frac{\kappa}{1+\kappa^2}\,exp(-\frac{\sqrt{2}\kappa z_t}{\tau})\,dz_t \\
&= \frac{\sqrt{2}}{\tau}\frac{\kappa}{1+\kappa^2}\int_{m_{d,q}}^{+\infty}\frac{-\tau}{\sqrt{2}\kappa}d(exp(-\frac{\sqrt{2}\kappa z_t}{\tau})) \\
&= -\frac{1}{1+\kappa^2}\left(exp(-\frac{\sqrt{2}\kappa(+\infty)}{\tau}) - exp(-\frac{\sqrt{2}\kappa\, m_{d,q}}{\tau})\right) \\
&= \frac{1}{1+\kappa^2}exp(-\frac{\sqrt{2}\kappa\, m_{d,q}}{\tau}) = \alpha
\end{aligned}$$

where $f^+(z_t)$ is the positive part of the $p.d.f.$ of ALD. Transforming this equation and setting $\tau = 1$, we can obtain the VaR for oil demand:

$$VaR_{d,t} = \mu + m_{d,q}\sigma_t = \mu - \frac{\sigma_t}{\sqrt{2}\kappa}\,ln(\alpha(1+\kappa^2)) \tag{20}$$

and further the CVaR for oil demand:

$$CVaR_{d,t} = E\left[y_t | y_t > VaR_{d,t}\right] = VaR_{d,t} + \frac{\sigma_t}{\sqrt{2}\kappa} \tag{21}$$

C. Derivation of the pdf of scaled ALD

Consider a random variable z follows the Asymmetric Laplace density function in equation (17) with mean and variance given by:[15]

$$E(z) = \theta + \frac{\tau}{\sqrt{2}}(\frac{1}{\kappa} - \kappa) \qquad\qquad Var(z) = \frac{\tau^2}{2}(\frac{1}{\kappa^2} + \kappa^2)$$

[15]More details of the mean and variance can refer to Kotz et al. (2001) for more details.

where $\tau = \sigma$ in our setting and is a constant, then we can transform z into another random variable ε_t by taking:

$$\varepsilon_t = \frac{z}{\sqrt{Var(z)}} \tag{22}$$

Taking partial derivatives of ε_t with respect to z, then we have:

$$dz = \sqrt{Var(z)}\, d\varepsilon_t = \frac{\tau}{\sqrt{2}} \frac{\sqrt{1+\kappa^4}}{\kappa}\, d\varepsilon_t \tag{23}$$

In the case $z \geq 0$ or $\varepsilon_t \geq 0$, by substituting (22) and (23) into density function (17), we are able to obtain:

$$\begin{aligned} Pr^+(\varepsilon_t) &= \int_0^{+\infty} \frac{\sqrt{2}}{\tau} \frac{\kappa}{1+\kappa^2} \frac{\tau}{\sqrt{2}} \frac{\sqrt{1+\kappa^4}}{\kappa}\; exp(\frac{-\sqrt{2}\kappa}{\tau} \frac{\tau}{\sqrt{2}} \frac{\sqrt{1+\kappa^4}}{\kappa}(\varepsilon_t - \theta))\, d\varepsilon_t \\ &= \int_0^{+\infty} \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\; exp(-\sqrt{1+\kappa^4}\,(\varepsilon_t - \theta))\, d\varepsilon_t \end{aligned} \tag{24}$$

Similarly, in the case $z < 0$ or $\varepsilon_t < 0$, it has:

$$\begin{aligned} Pr^-(\varepsilon_t) &= \int_{-\infty}^{0} \frac{\sqrt{2}}{\tau} \frac{\kappa}{1+\kappa^2} \frac{\tau}{\sqrt{2}} \frac{\sqrt{1+\kappa^4}}{\kappa}\; exp(\frac{\sqrt{2}}{\tau\kappa} \frac{\tau}{\sqrt{2}} \frac{\sqrt{1+\kappa^4}}{\kappa}(\varepsilon_t - \theta))\, d\varepsilon_t \\ &= \int_{-\infty}^{0} \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\; exp(\frac{\sqrt{1+\kappa^4}}{\kappa^2}\,(\varepsilon_t - \theta))\, d\varepsilon_t \end{aligned} \tag{25}$$

As a result, the *pdf* of SALD of random variable ε_t given σ_t can be written as:[16]

$$f(\varepsilon_t|\kappa, \theta, \sigma_t) = \begin{cases} \frac{\sqrt{1+\kappa^4}}{1+\kappa^2} \frac{1}{\sigma_t}\; exp(\frac{-\sqrt{1+\kappa^4}}{\sigma_t}\,(\varepsilon_t - \theta)) & \varepsilon_t \geq \theta \\ \frac{\sqrt{1+\kappa^4}}{1+\kappa^2} \frac{1}{\sigma_t}\; exp(\frac{\sqrt{1+\kappa^4}}{\kappa^2\sigma_t}\,(\varepsilon_t - \theta)) & \varepsilon_t < \theta \end{cases} \tag{26}$$

where κ is skewness parameter and σ_t is the time-varying volatility of return series.

[16]Note that parameter τ has been canceled out in this derivation.

D. Derivation of scaled ALD as an SMU

This part demonstrates the derivation of SALD as a scale mixture of $f_U(\varepsilon_t|\theta - \frac{\lambda\kappa^2\sigma_t}{\sqrt{1+\kappa^4}}, \theta + \frac{\lambda\sigma_t}{\sqrt{1+\kappa^4}})$ and $f_{Ga}(\lambda|2,1)$:

$$\begin{aligned} f(\varepsilon_t|\kappa,\theta,\lambda,\sigma_t) &= \int_0^\infty f_U(\varepsilon_t|\theta - \frac{\lambda\kappa^2\sigma_t}{\sqrt{1+\kappa^4}}, \theta + \frac{\lambda\sigma_t}{\sqrt{1+\kappa^4}}) \times f_{Ga}(\lambda|2,1)\, d\lambda \\ &= \int_0^\infty \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t}\frac{1}{\lambda} I(\theta - \frac{\lambda\kappa^2\sigma_t}{\sqrt{1+\kappa^4}} < \varepsilon_t < \theta + \frac{\lambda\sigma_t}{\sqrt{1+\kappa^4}})\, \lambda\, exp(-\lambda)\, d\lambda \end{aligned} \tag{27}$$

Consider two cases for random variable ε_t where (1): $\varepsilon_t > \theta - \frac{\lambda\kappa^2\sigma_t}{\sqrt{1+\kappa^4}}$ or equivalently $\lambda > \frac{-\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\kappa^2\sigma_t}$ and (2): $\varepsilon_t < \theta + \frac{\lambda\sigma_t}{\sqrt{1+\kappa^4}}$ or equivalently $\lambda > \frac{\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\sigma_t}$.

Case (1):

$$\begin{aligned} &\int_0^\infty \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t} I(\lambda > \frac{-\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\kappa^2\sigma_t})\, exp(-\lambda)\, d\lambda \\ =&\frac{-\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t}\int_{\frac{-\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\kappa^2\sigma_t}}^\infty exp(-\lambda)\, d(-\lambda) = \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t} exp(\frac{\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\kappa^2\sigma_t}) \end{aligned} \tag{28}$$

Since $\frac{-\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\kappa^2\sigma_t} > 0$, thus we have $\varepsilon_t < \theta$, which follows:

$$f^-(\varepsilon_t|\kappa,\theta,\sigma_t) = \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t} exp(\frac{\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\kappa^2\sigma_t}) \qquad \varepsilon_t < \theta$$

Case (2):

$$\begin{aligned} &\int_0^\infty \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t} I(\lambda > \frac{\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\sigma_t})\, exp(-\lambda)\, d\lambda \\ =&\frac{-\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t}\int_{\frac{\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\sigma_t}}^\infty exp(-\lambda)\, d(-\lambda) = \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t} exp(\frac{-\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\sigma_t}) \end{aligned} \tag{29}$$

Since $\frac{\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\sigma_t} \geq 0$, thus we have $\varepsilon_t \geq 0$, which follows:

$$f^+(\varepsilon_t|\kappa,\theta,\sigma_t) = \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t} exp(\frac{-\sqrt{1+\kappa^4}(\varepsilon_t-\theta)}{\sigma_t}) \qquad \varepsilon_t \geq \theta$$

As a result, it is demonstrated that the scaled Asymmetric Laplace density function of random variable ε_t:

$$f(\varepsilon_t|\kappa,\theta,\sigma_t) = \begin{cases} \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t}\ exp(\frac{-\sqrt{1+\kappa^4}}{\sigma_t}(\varepsilon_t-\theta)) & \varepsilon_t \geq \theta \\ \frac{\sqrt{1+\kappa^4}}{1+\kappa^2}\frac{1}{\sigma_t}\ exp(\frac{\sqrt{1+\kappa^4}}{\kappa^2\sigma_t}(\varepsilon_t-\theta)) & \varepsilon_t < \theta \end{cases} \tag{30}$$

can be replaced by an SMU distribution given by:

$$f(\varepsilon_t|\kappa,\theta,\lambda,\sigma_t) = \int_0^\infty f_U(\varepsilon_t|\theta - \frac{\lambda\kappa^2\sigma_t}{\sqrt{1+\kappa^4}}, \theta + \frac{\lambda\sigma_t}{\sqrt{1+\kappa^4}}) \times f_{Ga}(\lambda|2,1)\, d\lambda \tag{31}$$

E. Derivation of full conditional distributions

This part presents brief derivation of the full conditional distributions of model parameters and latent volatilities under the SMU of ALD.

- For parameter δ, we have:

$$\begin{aligned} f(\delta|\beta,\sigma_\eta^2,h,y) &\propto f(h_1|\delta,\beta,\sigma_\eta^2)\prod_{t=2}^{T} f(h_t|h_{t-1},\delta,\beta,\sigma_\eta^2) f_N(\mu_\delta,\sigma_\delta^2) \\ &= exp\left[-\frac{(h_1-\delta)^2(1-\beta^2)}{2\sigma_\eta^2} - \frac{\sum_{t=2}^{T}(h_t-\delta-\beta(h_{t-1}-\delta))^2}{2\sigma_\eta^2}\right] \\ &\left(\frac{1}{\sigma_\eta^2}\right)^{\frac{T}{2}} \frac{1}{\sqrt{2\pi}\sigma_\delta} exp\left[-\frac{(\delta-\mu_\delta)^2}{2\sigma_\delta^2}\right] \\ &\propto exp\Bigg\{-\frac{1}{2}\Bigg\{\underbrace{\left[\frac{1-\beta^2+(T-1)(1-\beta^2)}{\sigma_\eta^2}+\frac{1}{\sigma_\delta^2}\right]_{\text{A}}}\delta^2 \\ &-2\delta\underbrace{\left[\frac{h_1(1-\beta^2)+(1-\beta)\sum_{t=2}^{T}(h_t-\beta h_{t-1})}{\sigma_\eta^2}+\frac{\mu_\delta}{\sigma_\delta^2}\right]}_{\text{B}}\Bigg\}\Bigg\} \end{aligned} \tag{32}$$

Hence, we can obtain:

$$\delta|\beta,\sigma_\eta^2,h,y \sim N(\frac{B}{A},\frac{1}{A})$$

- Persistence parameter β ranges from -1 to 1 for stationarity, a beta prior distribution is assigned to $\beta^* = \frac{\beta+1}{2} \sim Be(a_\beta, b_\beta)$, then we have:

$$\begin{aligned} f(\beta^*|\delta,\sigma_\eta^2,h,y) &\propto f(h_1|\delta,\beta,\sigma_\eta^2)\prod_{t=2}^{T} f(h_t|h_{t-1},\delta,\beta,\sigma_\eta^2) f_{Be}(a_\beta,b_\beta) \\ &= exp\left[-\frac{(h_1-\delta)^2(1-\beta^2)}{2\sigma_\eta^2} - \frac{\sum_{t=2}^{T}(h_t-\delta-\beta(h_{t-1}-\delta))^2}{2\sigma_\eta^2}\right] \\ &\left(\frac{1}{\sigma_\eta^2}\right)^{\frac{T}{2}} \frac{\beta^{*(a_\beta-1)}(1-\beta^*)^{(b_\beta-1)}}{B(a_\beta,b_\beta)} \\ &\propto exp\left[\frac{\beta\sum_{t=2}^{T}(h_t-1)(h_{t-\delta}-\delta)}{\sigma_\eta^2} + \frac{\beta^2[(h_1-\delta)^2-\sum_{t=2}^{T}(h_{t-1}-\delta)^2]}{2\sigma_\eta^2}\right] \\ &(1+\beta)^{(a_\beta-1)}(1-\beta)^{(b_\beta-1)} \end{aligned} \tag{33}$$

where $B(\cdot,\cdot)$ is beta function with $B(a_\beta, b_\beta) = \frac{\Gamma(a_\beta)\Gamma(b_\beta)}{\Gamma(a_\beta+b_\beta)}$, and $\Gamma(\cdot)$ is gamma function.

- For parameter σ_η^2, we have:

$$\begin{aligned} f(\sigma_\eta^2|\delta,\beta,h,y) &\propto f(h_1|\delta,\beta,\sigma_\eta^2)\prod_{t=2}^{T} f(h_t|h_{t-1},\delta,\beta,\sigma_\eta^2) f_{IG}(a_\sigma,b_\sigma) \\ &= \frac{1}{\sqrt{2\frac{\sigma_\eta^2}{1-\beta^2}\pi}} exp\left[\frac{-(h_1-\delta)^2}{2\frac{\sigma_\eta^2}{1-\beta^2}}\right] \frac{1}{\sqrt{2\sigma_\eta^2\pi}} \\ &exp\left[\frac{-\sum_{t=2}^{T}(h_t-\delta-\beta(h_{t-1}-\delta))^2}{2\sigma_\eta^2}\right] \frac{b_\sigma^{a_\sigma}}{\Gamma(a_\sigma)} \sigma_\eta^{2(-a_\sigma-1)} exp\left(-\frac{b_\sigma}{\sigma_\eta^2}\right) \\ &\propto exp\left[-\frac{b_\sigma + \frac{1}{2}(h_1-\delta)^2(1-\beta^2) + \frac{1}{2}\sum_{t=2}^{T}(h_t-\delta-\beta(h_{t-1}-\delta))^2}{\sigma_\eta^2}\right] \\ &\left(\frac{1}{\sigma_\eta^2}\right)^{(a_\sigma+\frac{T}{2})+1} \end{aligned} \tag{34}$$

Therefore, we can obtain:

$$\sigma_\eta^2|\delta,\beta,h,y \sim IG(\hat{a}_\sigma, \hat{b}_\sigma)$$

where $\hat{a}_\sigma = a_\sigma + \frac{T}{2}$ and $\hat{b}_\sigma = b_\sigma + \frac{1}{2}(h_1-\delta)^2(1-\beta^2) + \frac{1}{2}\sum_{t=2}^{T}(h_t-\delta-\beta(h_{t-1}-\delta))^2$.

- For latent variables h_t, we have:

$$\begin{aligned}
f(h_t|h_{-t},\delta,\beta,\sigma_\eta^2,y) &\propto \; f(y|h_t,\delta,\beta,\sigma_\eta^2)f(h_t|h_{-t},\delta,\beta,\sigma_\eta^2) \\
&= \frac{1}{\frac{\lambda e^{h_t/2}}{\sqrt{1+\kappa^4}} + \frac{\lambda\kappa^2 e^{h_t/2}}{\sqrt{1+\kappa^4}}} \; \frac{1}{\sqrt{2\pi B^2}} \; exp\left[-\frac{(h_t-A)^2}{2B^2}\right] \\
&\propto e^{-\frac{h_t}{2}} \; exp\left[-\frac{1}{2}\left(\frac{h_t^2}{B^2} - 2h_t\frac{A}{B^2}\right)\right] \\
&= exp\Bigg\{ -\frac{1}{2}\left[\underbrace{\frac{1}{B^2}}_{\text{C}} \, h_t^2 - 2h_t \underbrace{\left(\frac{A}{B^2} - \frac{1}{2}\right)}_{\text{D}}\right]\Bigg\}
\end{aligned} \tag{35}$$

where

$$A = \delta + \frac{\beta[(h_{t-1}-\delta) + (h_{t+1}-\delta)]}{1+\beta^2}, \qquad B^2 = \frac{\sigma_\eta^2}{1+\beta^2}$$

is the mean and variance of Normal density function $f_N(h_t|A,B^2)$, which has equality:

$$f(h_t|h_{-t},\delta,\beta,\sigma_\eta^2) = f(h_t|h_{t-1},h_{t+1},\delta,\beta,\sigma_\eta^2) = f_N(h_t|A,B^2)$$

Hence, it can be shown that:

$$h_t|h_{-t},\delta,\beta,\sigma_\eta^2,y \sim N(\frac{D}{C},B^2) \quad or \quad N(A-\frac{B^2}{2},B^2) \tag{36}$$