Coalitional Stability in a Class of Social Interactions Games*

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Abstract

In this paper, we define additive dyadic social interactions games (ADG), in which each player cares not only about the selected action, but also about interactions with other players, especially those who choose the same action. This class of games includes alliance formation games, network games, and discrete choice problems with network externalities. While it is known that games in the ADG class admit a pure strategy Nash equilibrium that is a maximizer of the game's potential, the potential approach does not always apply if all coalitional deviations are allowed. We then introduce a novel notion of a strong landscape equilibrium, which relies on a limited scope of coalitional deviations. We show the existence of a *strong landscape equilibrium* for a class of basic additive dyadic social interactions games ($BADG$), even though a strong Nash equilibrium may fail to exist. Somewhat surprisingly, a potential-maximizing strong landscape equilibrium is not always a strong Nash equilibrium even if the set of the latter is nonempty. We also provide applications and extensions of our results.

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1 Introduction

In this paper we conduct an equilibrium analysis of games with social interactions where actions taken by each player affect not only their own outcome but also that of other players. [1] The impact of social interactions is evident in a wide range of individual decisions. These include making residential choices (Schelling, 1969), deciding to become a member of a club (Dixit, 2003), quitting smoking (Harris and Lopez-Varcarcel, 2004), engaging in criminal activities (Glaeser et al. 1996), joining a side in international conflict (Altfeld and Bueno de Mesquita, 1979, Axelrod and Bennett, 1993), or considering acquiring a non-native language (Brock et al., 2025). A collective action problem has been examined by Schelling (1978), who considered a setting where the participation of an individual in a protest action depends on the fraction of the population engaged in the action.

More specifically, we consider a society with n individuals and a set of m feasible actions. When an individual i selects an action k, her payoff is given by the sum of n terms:

$$v_k^i + \sum_{j \neq i} v_{kl}^{ij}, \tag{1}$$

where the first term v_k^i represents the intrinsic payoff derived by i from selecting action k, and v_{kl}^{ij} is the benefit to player i from her interaction with j when j selects action l. We further impose a standard symmetry assumption: for each pair of players i and j, and two actions k and l, the benefit that i choosing k derived from j choosing l, is equal to the dyadic interaction payoff of j choosing k derives from i when she selects l.

Note that if the payoff contained only the first term in (1), every individual would simply choose an action that maximizes her intrinsic payoff. However, in general, the presence of $n-1$ other terms that represent the benefit to individual i derived from her social interactions with other players could alter i's action choice. To reflect the additive and dyadic nature of our payoff structure that relies on pairwise interactions, we shall call the class of these social interactions games *Additive Dyadic Games* and refer to it as *ADG*.

We would like to point out that most of the contributions cited in this paper address the existence and characterization of pure-strategy Nash equilibria for several subclasses of *ADG* (Ballester et al., 2006; Axelrod and Bennett, 1993; Soetevent and Kooreman, 2007), and even for the entire set of *ADG* (Le Breton and Weber, 2011). In many cases, the existence proof relies on the construction of a *potential function* on the set of profiles of actions whose local maximizers are Nash equilibria of the game (Monderer and Shapley, 1996). However, the number of pure-strategy Nash equilibria could be very large. For example, Dower et al. (2024) identified 86 pure-strategy Nash equilibria in an *ADG* game with 46 players and two feasible actions.

[1] Thus, while players obviously care about the chosen action, they are also mindful of externalities imposed by other players on a bilateral basis, including the composition of the group of players choosing the same action.

Thus, following Monderer and Shapley (1996), several researchers use the potential-maximizing Nash equilibrium as a natural refinement selection from the equilibrium set. Another refinement route utilized in this paper is based on allowing action coordination, which plays an important role in social externality settings.

As we alluded to above, the *ADG* class contains the family of network games examined in Ballester et al. (2006). They assume that a player's payoff function is quadratic:

$$u^i(x^1,...,x^n) = a^i x^i + \frac{1}{2}b(x^i)^2 + \sum_{j\neq i} p^{ij} x^i x^j,$$

where $x^i \in \mathbb{R}_+$ is player i's effort level, and a^i, b, and p^{ij} are parameters. Here, the matrix $(p^{ij})_{i,j=1,...,n}$ is a *symmetric* matrix that describes the strength of social interaction for each pair of players i and j. The sign of p^{ij} can be positive or negative. Ballester et al. (2006) show that if an interior Nash equilibrium of this game exists, it is unique and is described by a set of linear equations.[2] Now, letting actions k and l represent the levels of effort $e_k, e_l \in \mathbb{R}_+$ exerted by players i and j choosing actions k and l, respectively, we can treat their network game as an *ADG* with payoff function (1) with:

$$\begin{aligned} v^i_k &= a^i e_k + \frac{1}{2}b e_k^2, \\ v^{ij}_{kl} &= p^{ij} e_k e_l. \end{aligned}$$

Thus, the fact that their game belongs to the class of *ADG* implies that the existence of a Nash equilibrium of the game does not rely on the continuous single-dimensional action space.

Another subclass of *ADG* is represented by a discrete choice problem, where social interactions are generated by the impact of players choosing the same alternative (see Brock and Durlauf, 2001, 2002, and Soetevent and Kooreman, 2007), who study a two-stage game on networks that assigns links between players. In their setting, action k corresponds to an activity (alternative) each player can choose, and the dyadic social interaction v^{ij}_{kl} is *local* in the sense that it is in effect only within the same activity:

$$v^{ij}_{kl} = \begin{cases} p^{ij} & \text{if } k = l; \\ 0 & \text{otherwise.} \end{cases}$$

where the matrix $(p^{ij})_{i,j=1,...,n}$ is a *propensity matrix* that describes how well each pair i and j get along. Again, p^{ij} can be positive or negative, but is symmetric $p^{ij} = p^{ji}$. Thus, in selecting their activities, players care not only about preferences over activities, but also about the composition of those players who choose the same activities. In the one-player setting with $n = 1$, our model corresponds to the celebrated random utility model used extensively in the standard econometrics of discrete choice models (McFadden, 1981, Manski, 2000). In that respect, the *ADG* class can just be viewed

[2] Under further assumptions on the parameters, Ballester et al. (2006) explore the properties of equilibrium as a function of the matrix $(p^{ij})_{i,j=1,...,n}$.

as an n-player extension of the random utility model with Nash equilibrium and its refinements replacing utility maximization.

Last but not least, note that various facets of international alliances such as bloc formation in a conflict (Altfeld and Bueno de Mesquita, 1979) or voting behavior in the UN Assembly (Russett, 1966) constitute potential applications of the *ADG* framework. In fact, Axelrod and Bennett (1993), and Bennett (2000) modeled an international alliance game as a *local ADG*, with players choosing one of the two existing blocs, and examined the stability of nations' alliance structure. The key ingredient in Axelrod and Bennett's landscape theory of aggregation is the *propensity matrix* $(p^{ij})_{i,j=1,\ldots,n}$ which describes the sign and magnitude of the dyadic social externality between each pair of players i and j. The sign of a matrix entry again indicates the nature of the externality (negative for foes and positive for allies) and is symmetric, while the absolute value represents the intensity of this negative or positive interaction.

While the reliance on potential functions has been useful in the examination of Nash equilibria, an attempt to apply the maximization of the potential function to coalitional deviations has been unsuccessful. As our Example 2 indicates, the potential maximization is inconsistent with coalitional deviations. We offer a two-part approach to rectify this inconsistency issue. First, noting that a strong Nash equilibrium with no restriction on coalitional deviations (Aumann 1959) may fail to exist for games in *ADG* (Banerjee et al., 2001, and Example 2 in this paper), we define the notion of a *strong landscape equilibrium*, which is immune to limited coalitional deviations with some restrictions based on communication/coordination costs across groups, and is weaker than a strong Nash equilibrium. In addition, we identify subclasses of the *ADG* that admit our equilibrium selection. These subclasses satisfy various uniformity conditions on individuals' intrinsic preferences and the values of their social interactions. It turns out these two modifications allow us to demonstrate the existence of a strong landscape Nash equilibrium for a wide class of games.

This paper is organized as follows. Following the introduction, in Section 2 we provide a review of the relevant literature. In Section 3, the existence of Nash equilibrium within the *ADG* class is discussed. In Section 4, we introduce various equilibrium concepts (including a novel notion of a strong landscape equilibrium) are introduced. We demonstrate that the distinction between the three sets of landscape, strong landscape and strong equilibria is not vacuous, and, in general, they do not coincide (Examples 2 and 3). In Section 5, we focus on basic additive social interactions games and prove our main result on the existence of a strong landscape equilibrium within this class of games (Theorem 2). In Section 6 we impose another restriction, *neutrality* ($v^i_k = 0$ for all i and all k) on the game to show that there is a strongly Pareto optimal strong landscape equilibrium (Proposition 1), although there may not be a Pareto efficient Nash equilibrium without neutrality (Example 4). We also show that a potential-maximizing strong landscape equilibrium is not necessarily a strong Nash equilibrium even if there is a strong Nash equilibrium (Example 5), and a game may have two Pareto-ranked strong landscape equilibria (Example 6) even under neutrality. In Section 7 we consider a class of basic games that allow for an outside option. While the existence of a strong landscape equilibrium is no longer guaranteed (Example 8), the existence of a strong landscape equilibrium is shown under neutrality (Proposition 2). Moreover, there exists a strong Nash equilibrium when dyadic social interactions are non-negative and $m = 2$ (Proposition 3). However, unlike in Dower et al. (2024), the potential-maximizing strategy profile is not necessarily a strong Nash equilibrium with two feasible actions (Example 9). Section 8 introduces population externalities. Example 10 shows that a strong landscape equilibrium may fail to exist within the class $BADG$, but under a special form of population externality the existence of a strong landscape equilibrium can be recovered (Proposition 5). In Section 9 we address the question of when a game admits an ADG payoff representation. This question is a difficult challenge and we offer a partial answer (Proposition 6), whereas Example 11 illustrates the difficulties in assuring the symmetry in this setting. Finally, Example 12 demonstrates that a monotone transformation of a game in the $NBADG$ class, may alter the set of potential maximizers. Section 10 concludes the paper with some remarks, including an application to a class of matching problems with peer interactions.

2 The Literature Review

One of the primary applications of our ADG was the study of international alliances by Axelrod and Bennett (1993), who analyzed the alliance game where players face a choice of joining one of the two existing blocs. The Axelrod-Bennett analysis is based on their landscape theory of aggregation and uses the energy landscape (the inverse of the potential function) as a theoretical tool to evaluate the level of stability of nations' alliance structure. They apply their model to the empirical study of the alignment of 17 European nations with the Allies and Axis during World War II. To do so, Axelrod and Bennett (1993) presented a method for the empirical identification of the propensity matrix. Following the Axelrod-Bennett contribution, Florian and Galam (2000) generalized the landscape theory to more than two blocs and extended their analysis to the break-up of Yugoslavia more than thirty years ago. Kijima (2001) applied this extended framework to analyze alliance formation in the civil aviation industry.

It is worth pointing out that the notion of *dyadic affinities* (the terminology coined by Kuperman et al., 2006), which is central in the ADG framework, is a key feature of academic research in international politics. As we alluded to earlier, Bueno de Mesquita (1975, 1981) was the first to transform the notion of structural affinity into an empirical measure via the similarity of alliance portfolios. Signorino and Ritter (1999) offer an alternative way of deriving the propensity matrix. Kuperman et al. (2006) discuss the properties and limitations of these two measures of structural affinity and offer an alternative one.

Additive dyadic social interactions games have been used for empirical research of social interactions under the common term v^{ij} in (1). For every player i, v^i_k is the sum of a deterministic part observed by all players and a player-specific random term, and social interactions depend on n_k, which is the number of players choosing action k. In Brock and Durlauf (2001, 2002), players are assumed to make their action choice based on their expected memberships of actions, $(n_k)_{k=1}^m$, computed from the distributions of player-specific random terms. Brock and Durlauf (2001, 2002) and Blume et al. (2015) analyze the equilibrium of the game with rational expectations. In contrast, Soetevent and Kooreman (2007) assume that player-specific random terms are common knowledge and use Nash equilibrium as a solution concept when $m = 2$. They prove existence of a Nash equilibrium, provide bounds on the number of equilibria, and explore the econometric side of the model, offering an empirical illustration.

A number of papers analyze network games with dyadic social interactions, including the quadratic-utility continuous action choice game of Ballester et al. (2006) and Bramoullé and Kranton (2007). Bramoullé et al. (2014) show that there is a potential function for this game and examine various applications of social interactions games, including R&D competition and societal crime patterns.

Some papers examine equilibrium stability and the robustness of equilibria as a refinement concept. Ui (2001) provides support for such an equilibrium selection by showing its robustness in the sense of Kajii and Morris (1997). Carbonell-Nicolau and McLean (2014) show that the set of potential maximizers contains a Kohlberg and Mertens (1986) stable set of pure-strategy Nash equilibria. Blume (1993) shows that the potential maximizers correspond to stochastically stable states under log-linear dynamics such as the logit choice rule. Bramoullé et al. (2014) demonstrate the stability of the potential-maximizing Nash equilibrium.

Newton and Sercombe (2020) consider coalitional deviations as well as unilateral deviations in analyzing the diffusion of an innovation on a network using a two-action symmetric coordination game with a potential function. They compare the graph-theoretic properties under which potential-maximizing strategy profiles are evolutionarily stable with unilateral (stochastic) and coalitional (coordinated) deviations. Le Breton et al. (2021) extend the analysis of Axelrod and Bennett (1993) by allowing for landscape coalitional deviations to show that the potential-maximizing Nash equilibrium is immune to *landscape* deviations and is strongly Pareto efficient. Dower et al. (2024) analyze the two paths (Western and Eastern) of the post-war development strategies chosen by African countries and estimate a propensity matrix, similarly to Axelrod and Bennett (1993). They show the existence of a strong Nash equilibrium and compute a potential-maximizing strong Nash equilibrium of the game between African countries. They identify a large number of Nash equilibria, only one of which is a strong Nash equilibrium and a potential maximizer.

3 Potential Functions and Nash Equilibria in the Additive Dyadic Games

Here, we introduce a class of additive dyadic social interactions games. There are a finite set of players $N = \{1, 2, ..., n\}$, and a finite set of feasible actions $M = \{1, 2, ..., m\}$, where each player in N chooses an action s^i from M. The resulting strategy profile $\mathbf{s} = (s^i)_{i \in N}$ partitions players over the set of actions $\mathbf{G}(\mathbf{s}) = (G_\ell(\mathbf{s}))_{\ell \in M}$, where $G_\ell(\mathbf{s}) \equiv \{j \in N : s^j = \ell\}$ is the set of players who chose the same action ℓ.

Let $p^{ij}_{k\ell} \in \mathbb{R}$ denote player i's dyadic propensity from player j when player i chooses action k and player j chooses action ℓ, and let $\sigma^j > 0$ be an influence parameter—it indicates the impact of player j on others.

The propensities described here are represented by the following matrices:

$$P_{kk} = \begin{pmatrix} p^{11}_{kk} & \cdots & p^{1i}_{kk} & \cdots & p^{1j}_{kk} & \cdots & p^{1n}_{kk} \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ p^{i1}_{kk} & \cdots & p^{ii}_{kk} & \cdots & \underline{p^{ij}_{kk}} & \cdots & p^{in}_{kk} \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ p^{j1}_{kk} & \cdots & \underline{p^{ji}_{kk}} & \cdots & p^{jj}_{kk} & \cdots & p^{jn}_{kk} \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ p^{n1}_{kk} & \cdots & p^{ni}_{kk} & \cdots & p^{nj}_{kk} & \cdots & p^{nn}_{kk} \end{pmatrix}$$

and

$$P_{k\ell} = \begin{pmatrix} p^{11}_{k\ell} & & & \\ & \ddots & \boxed{p^{ji}_{k\ell}} & \\ & \mathbf{p}^{ij}_{k\ell} & \ddots & \\ & & & p^{nn}_{k\ell} \end{pmatrix}; \;\; \& \;\; P_{\ell k} = \begin{pmatrix} p^{11}_{\ell k} & & & \\ & \ddots & \mathbf{p}^{ji}_{\ell k} & \\ & \boxed{p^{ij}_{\ell k}} & \ddots & \\ & & & p^{nn}_{\ell k} \end{pmatrix}$$

Here, P_{kk} is a symmetric matrix for any $k \in M$, but across action propensity matrices $P_{k\ell}$ and $P_{\ell k}$ are not necessarily symmetric (meaningless diagonal terms satisfy $p^{ii}_{k\ell} = p^{ii}_{\ell k} = 0$).

Then the payoff specification (1) for player i who chooses an action $k \in M$ under strategy profile $\mathbf{s}$ yields

$$u^i_k(\mathbf{G}) = v^i_k + \sum_{\ell \in M} \sum_{j \in G_\ell} p^{ij}_{k\ell} \sigma^j = v^i_k + \sum_{j \in G_k} p^{ij}_{kk} \sigma^j + \sum_{\ell \in M \setminus \{k\}} \sum_{j \in G_\ell} p^{ij}_{k\ell} \sigma^j,$$

where $v^i_k \in \mathbb{R}$ is, to recall, an intrinsic benefit derived by player i from choosing action k.

Note that if we multiply the right-hand of expression (2) by σ^i, then we obtain a new game that is strategically equivalent to the original game.

We further impose the straightforward normalization conditions: (i) $p^{ii}_{kk} = 0$ for all $i \in N$ and all $k \in M$, and (ii) $p^{ii}_{k\ell} = 0$ for all $i \in N$ and all $k, \ell \in M$ with $k \neq \ell$.

Note the symmetry condition $p_{k\ell}^{ij} = p_{\ell k}^{ji}$ for all $i, j \in N$ and all $k, \ell \in M$, simply means that propensities are symmetric as the dyadic interactions are symmetric.

A game that belongs to this class is called an **additive dyadic game**, and the set of additive dyadic games is denoted by ADG.

While Le Breton and Weber (2011) have shown that every game in the ADG class admits a Nash equilibrium in pure strategies, in order to make our presentation self-contained, we present here a shorter proof of the existence result and underline the importance of the potential functions technique (Monderer and Shapley, 1996)) whose local maximizers are Nash equilibria of the game. For this end, we define the following function, which will be shown to be a weighted potential of the game $\Gamma \in ADG$.

$$\mathbb{F}(\mathbf{s}) \equiv \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} \sigma^j v_\ell^j + \frac{1}{2} \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} \sum_{\ell' \in M} \sum_{j' \in G_{\ell'}(\mathbf{s})} \sigma^j p_{\ell\ell'}^{jj'} \sigma^{j'}. \tag{2}$$

To see that $\mathbb{F}(\mathbf{s})$ is a weighted potential, we let $\mathbf{s} = (s^j)_{j \in N}$ with $s^j = k$, and consider a potential of a reduced game without player i and strategy profile $\mathbf{s}^{-i}$. Then, we have

$$\mathbb{F}(\mathbf{s}) - \mathbb{F}(\mathbf{s}^{-i}) = \sigma^i v_k^i + \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} \sigma^i p_{k\ell}^{ij} \sigma^j,$$

since there are two cases $k = \ell$ and $k = \ell'$ with $p_{k\ell}^{ij} = p_{\ell k}^{ji}$ (and $p_{kk}^{ii} = 0$). Let $\tilde{\mathbf{s}} = (\tilde{s}^i, \mathbf{s}^{-i})$ with $\tilde{s}^i = h$. Similarly we have

$$\mathbb{F}(\tilde{\mathbf{s}}) - \mathbb{F}(\mathbf{s}^{-i}) = \sigma^i v_h^i + \sum_{\ell \in M} \sum_{j \in G_\ell(\tilde{\mathbf{s}})} \sigma^i p_{h\ell}^{ij} \sigma^j.$$

This implies

$$\mathbb{F}(\tilde{\mathbf{s}}) - \mathbb{F}(\mathbf{s}) = \sigma^i v_h^i + \sum_{\ell \in M} \sum_{j \in G_\ell(\tilde{\mathbf{s}})} \sigma^i p_{h\ell}^{ij} \sigma^j - \sigma^i v_k^i - \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} \sigma^i p_{k\ell}^{ij} \sigma^j.$$

If any player $i \in N$ switches her strategy from $s^i = k$ to $\tilde{s}^i = h$ unilaterally, then we have

$$\Delta u^i = u^i(\tilde{\mathbf{s}}) - u^i(\mathbf{s}) = v_h^i + \sum_{\ell \in M} \sum_{j \in G_\ell(\tilde{\mathbf{s}})} p_{h\ell}^{ij} \sigma^j - v_k^i - \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} p_{k\ell}^{ij} \sigma^j,$$

and we conclude

$$\mathbb{F}(\tilde{\mathbf{s}}) - \mathbb{F}(\mathbf{s}) = \sigma^i \Delta u^i.$$

Therefore, $\mathbb{F}(\mathbf{s})$ is a weighted potential of the game Γ. In contrast with the weighted Benthamite social welfare function, the second term of $\mathbb{F}(\mathbf{s})$ has the coefficient $\frac{1}{2}$. That is, when player i switches from k to h to join player j, the i's net welfare gain from the relation with a player j is the same as welfare gain derived by player j from the relation to i. The following is a variation of the result by Le Breton and Weber (2011).

Theorem 1: (Le Breton and Weber, 2011). Every game $\Gamma \in ADG$ admits a Nash equilibrium.[3]

Proof. Let $\bar{\mathbf{s}} \in \arg\max_{\mathbf{s}} \mathbb{F}(\mathbf{s})$. Suppose that $\bar{\mathbf{s}}$ is not a pure strategy Nash equilibrium. Then, there is player $i \in N$ and strategies $k \neq h$ such that $s^i = k$ and $\tilde{s}^i = h$ with $\Delta u^i = u^i(\tilde{s}^i, \bar{\mathbf{s}}^{-i}) - u^i(\bar{\mathbf{s}}) > 0$. However, this implies $\mathbb{F}(\tilde{s}^i, \bar{\mathbf{s}}^{-i}) > \mathbb{F}(\bar{\mathbf{s}})$, a contradiction.$\square$

Note that Theorem 1 covers the finite versions of the existence results in Ballester et al. (2006) and Bramoullé et al. (2014). Moreover, it is more general than the equilibrium result in Axelrod and Bennett (1993). While Brock and Durlauf (2001, 2002) use a different equilibrium notion, Soetevent and Kooreman (2007) Nash equilibrium existence result in that setting is also covered by Theorem 1.

It is important to point out that the relationship between the set of potential maximizers and the set of equilibria breaks down if we allow for coalitional deviations rather than individual switches of action. It is shown by the following example:

Example 1. Let $N = \{1, 2, 3, 4\}$ and $M = \{a, b\}$ with $v^i_\ell = 0$ for all $i \in N$ and all $\ell \in M$. Let game $\Gamma \in ADG$ be local: i.e., off-diagonal propensity matrices are $P_{ab} = P_{ba} = \mathbf{0}$. We set $\sigma^i = 1$ for all $i \in N$. Let

$$P_{aa} = \begin{pmatrix} 0 & 1.5 & 0 & 0 \\ 1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 \\ 0 & 0 & 1.5 & 0 \end{pmatrix} \qquad P_{bb} = \begin{pmatrix} 0 & 3 & 0 & -1 \\ 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 3 \\ -1 & 0 & 3 & 0 \end{pmatrix}$$

In this case, potential-maximizing allocations are $\mathbf{s} = (a, a, b, b)$ (or $\mathbf{s}' = (b, b, a, a)$). Consider a coalitional deviation by $\{1, 2\}$ moving to action b. With this deviation, players 1 and 2 improve their payoffs from 1.5 to 2 each, but the value of potential goes down from $\frac{1}{2}\{(1.5 + 1.5) + (3 + 3)\} = 4.5$ to $\frac{1}{2}\{2 + 2 + 2 + 2\} = 4.$$\blacksquare$

Thus, in order to guarantee the existence of equilibria immune against coalitional deviations, we need to limit the scope of permissible deviations and/or restrict the subset of the ADG games under consideration. It is done in the following section.

4 Coalitional Deviations and Subclasses of the ADG

In this section, we first identify several subclasses of ADG, We then define various equilibrium concepts that allow for coalitional deviations.

Definition 1.

[3]Since in our paper we refer only to pure strategies Nash equilibria, without any confusion, we will use the term of Nash equilibrium rather than Nash equilibrium in pure strategies.

1. A game $\Gamma \in ADG$ is **local** if $P_{k\ell} = (\mathbf{0})_{n\times n}$ for all $k, \ell \in M$ with $k \neq \ell$: i.e., social interactions take place only within groups of players who chose the same action.

2. A game $\Gamma \in ADG$ is **uniform** if diagonal propensity matrices are the same for all actions: i.e., $P_{kk} = P$ for all $k \in M$.

3. A game Γ is **basic** if it is *local* and *uniform*. The set of basic games is denoted by $BADG$.

4. A game Γ is **neutral** if it is *basic*, and, moreover, $v_k^i = 0$ for all $i \in N$ and all $k \in M$. The set of neutral games is denoted by $NBADG$.

Naturally, we have the following inclusion

$$NBADG \subset BADG \subset ADG.$$

We introduce some solution concepts for our games. First, a strategy profile is a **Nash equilibrium** if and only if for all $i \in N$, $u^i(\mathbf{s}) \geq u^i(\tilde{s}^i, \mathbf{s}^{-i})$ holds for all $\tilde{s}^i \in M$. While Nash equilibrium is based on individual players' unilateral deviations is the most fundamental solution concept, we are interested in exploring the framework that allows for coalitional deviations.

For every two strategy profiles $\mathbf{s}$ and $\tilde{\mathbf{s}}$, denote by $S(\mathbf{s}, \tilde{\mathbf{s}})$ the set of players who choose different strategies at $\mathbf{s}$ and $\tilde{\mathbf{s}}$, i.e.,

$$S(\mathbf{s}, \tilde{\mathbf{s}}) \equiv \left\{ i \in N : \tilde{s}^i \neq s^i \right\}.$$

A coalitional deviation from a strategy profile $\mathbf{s}$ via $\tilde{\mathbf{s}}$ is **strictly improving** if $u^i(\tilde{\mathbf{s}}) > u^i(\mathbf{s})$ for all $i \in S(\mathbf{s}, \tilde{\mathbf{s}})$. Without placing any restrictions on the set of coalitional deviations, we obtain the notion of strong Nash equilibrium (Aumann 1959).

Definition 2. A strategy profile $\mathbf{s}$ is a **strong Nash equilibrium** if there is no strictly improving coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}})$. The set of all strong Nash equilibria is denoted by SNE.

While being an attractive solution concept, a strong Nash equilibrium may often fail to exist for $BADG$ games, too.[4]

Example 2. Let $N = \{1, 2, 3\}$ and $M = \{a, b, c, d\}$. Suppose that game $\Gamma \in BADG$ has the following propensity matrix,

$$P = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & -2 \\ 1 & -2 & 0 \end{pmatrix}$$

[4]While Banerjee et al. (2001) show that SNE may fail to exist even within the $NBADG$ class. our simple Example 2 clearly illustrates the type of coalitional deviations that rule out the SNE existence.

and intrinsic preferences over actions,

$k \in M$	a	b	c	d
v_k^1	0	-10	-3.5	-10
v_k^2	-10	0	-0.5	-10
v_k^3	-10	-10	0	-0.5

We set $\sigma^i = 1$ for all $i \in N$. Players 1, 2, and 3 will choose their actions from $\{a, c\}$, $\{b, c\}$, and $\{c, d\}$, respectively. Player 1 may choose c only when both players 2 and 3 choose c. The following profiles are candidates for a strong Nash equilibrium.

	u^1	u^2	u^3	potential
(a, b, c)	0	0	0	0
(c, c, c)	0.5	0.5	-1	-2
(c, c, d)	-0.5	2.5	-0.5	-1.5
(a, c, d)	0	-0.5	-0.5	-1
(a, b, d)	0	0	-0.5	-0.5

Notice that there is a unique Nash equilibrium $\mathbf{s} = (a, b, c)$. However, it is not a strong Nash equilibrium, since players 1 and 2 will jointly move to c. Thus, a strong Nash equilibrium does not exist. The reason is as follows. At strategy profile $\mathbf{s}$, players 1 and 2 are choosing different actions, but they both shift to c. This coordinated action generates benefits which are internal for players 1 and 2, where no outsider benefits from this move. Thus, the change in the value of our potential function caused by this coalitional deviation does not corresponds to the joint benefit of the coalition.■

The above example shows that a coalitional deviation that involves coordinated moves by two players choosing different actions in the original strategy (action) profile, a potential function approach may threaten the existence of an equilibrium. Thus, we will limit the scope of coalitional deviations. Let $\mathbf{s}$ and $\tilde{\mathbf{s}}$ be two different profiles of actions, and k and h be two different actions. Let

$$S_{kh}(\mathbf{s}, \tilde{\mathbf{s}}) \equiv \left\{ i \in S(\mathbf{s}, \tilde{\mathbf{s}}) : s^i = k \text{ and } \tilde{s}^i = h \right\}$$

be the set of players who switched their strategy from k to h at $\mathbf{s}$. Clearly, the collection of (not necessarily nonempty) sets $\{S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})\}_{k,h \in M, k \neq h}$ is a partition of $S(\mathbf{s}, \tilde{\mathbf{s}})$, which is the set of all players whose chose different strategies at $\mathbf{s}$ and $\tilde{\mathbf{s}}$.

We consider the following two types of coalitional deviations:

Definition 3. Two types of coalitional deviations are defined as:

1. We say that a coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}}) \equiv \cup_{k \neq h} S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})$ is of **Type 1** if there is at most single k with $|S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})| > 0$. This condition means that if there are players who switch their action, they choose the same action k under $\mathbf{s}$. A strategy profile $\mathbf{s}$ is a **landscape equilibrium** if it is immune to all coalitional deviations of Type 1. The set of those equilibria is denoted by LE.

2. We say that a coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}}) \equiv \cup_{k \neq h} S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})$ is of **Type 2** if for any $h \in M$, there is at most one single k with $|S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})| > 0$. That is, if there are players who choose newly adopted action h under $\tilde{\mathbf{s}}$, all such players must have been choosing the same action k under $\mathbf{s}$. A strategy profile $\mathbf{s}$ is a **strong landscape equilibrium** if $\mathbf{s}$ is immune to all deviations of Type 2. The set of those equilibria is denoted by SLE.

The requirements imposed by coalitional deviations of Types 1 and 2 can be illustrated on a directed graph over the set of actions M. The changes of actions by the coalition members of deviation $S(\mathbf{s}, \tilde{\mathbf{s}})$ are summarized by a directed graph on action set M, $(M, \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}}))$, such that $kh \in \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}})$ if and only if $S_{kh}(\mathbf{s}, \tilde{\mathbf{s}}) \neq \emptyset$. The definition of a Type 1 deviation implies that $\mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}})$ is an out-tree that stems out from a single node k where the branches are restricted to the unit length only. In contrast, Type 2 condition says that each component of $\mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}})$ is a cycle, an out-tree, or a cycle with out-trees stemming out of it.[5] Thus, the Type 2 deviation covers much wider class of directed graphs, thus, yields allows for a wider range of coalitional deviations than that of Type 1. It is easy to see that the unique NE of Example 2 is also LE and SLE, while SNE does not exist.

The deviations of Type 1 are allowed only for a group of players who choose the same action at the original profile $\mathbf{s}$. The deviations of Type 2 are less restrictive and require that only those members of the deviating group who choose the same action at $\tilde{\mathbf{s}}$, also share the same action at $\mathbf{s}$. Thus, the set of landscape equilibria immune to Type 1 deviations contains the set of strong landscape equilibria immune to Type 2 deviations. We have the following relationship between the sets of equilibria: Naturally, we have the following inclusion:

$$SNE \subset SLE \subset LE \subset NE,$$

where NE denotes the set of pure strategies Nash equilibria. Example 2 shows that the sets SNE and SLE do not always coincide. Example 3 demonstrates that, in general, SLE, LE, and NE are different from each other.

Example 3. Let $N = \{1, 2, 3, 4, 5\}$ and $M = \{a, b, c\}$. Suppose that in game $\Gamma \in BADG$ with $\sigma^i = 1$ for all $i \in N$, players' payoff information is summarized by the following table and propensity matrix P:

k	a	b	c
v_k^1	0	1.5	-10
v_k^2	0	1.5	-10
v_k^3	0	-10	0.5
v_k^4	-10	0	-10
v_k^5	0.5	-10	0

$$P = \begin{pmatrix} 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{pmatrix}$$

[5] A path $\{(k_1, k_2), (k_2, k_3), ..., (k_{q-1}, k_q)\}$ can form a trivial cycle by including (k_q, k_1) with $S_{k_q k_1}(\mathbf{s}, \tilde{\mathbf{s}}) = \emptyset$.

The relevant strategy profiles and their payoff vectors are:

i	1	2	3	4	5		potential
$\mathbf{s}^1 = (a, a, a, b, c)$	1	1	0	0	0	NE	1
$\mathbf{s}^2 = (b, a, a, b, c)$	0.5	0	0	-1	0	no	0.5
$\mathbf{s}^3 = (a, a, c, b, a)$	0	0	0.5	0	-1.5	no	-1
$\mathbf{s}^4 = (b, b, a, b, c)$	1.5	1.5	0	-2	0	LE	2
$\mathbf{s}^5 = (b, b, c, b, a)$	1.5	1.5	0.5	-2	0.5	$SLE = SNE$	3
$\mathbf{s}^6 = (b, a, a, b, a)$	0.5	-1	-0.5	-1	-0.5	no	-3

Note that players 1 and 2 are symmetric. In this game, there are three Nash equilibria, $\mathbf{s}^1$, $\mathbf{s}^4$, and $\mathbf{s}^5$. From Nash equilibrium $\mathbf{s}^1$, players 1 and 2 can move together to b through a Type 1 coalitional deviation, generating $\mathbf{s}^4$. Although player 4 is negatively affected by the deviation by players 1 and 2, the new strategy profile $\mathbf{s}^4$ is immune to any further Type 1 coalitional deviations. Thus, $\mathbf{s}^4$ is a landscape equilibrium. However, from $\mathbf{s}^4$, a Type 2 coalitional deviation composed by players 3 and 5 swapping their actions generates $\mathbf{s}^5$, and $\mathbf{s}^5$ is a strong landscape equilibrium. Note that $\mathbf{s}^5$ can be brought directly from $\mathbf{s}^1$ by a three-group Type 1 coalitional deviation involving players 1, 2, 3, and 5 as well. As we saw earlier, the values of the potential of the game for $\mathbf{s}^1$, $\mathbf{s}^4$, and $\mathbf{s}^5$ are 1, 2, and 3, respectively. Thus,Type 2 coalitional deviations allow the value of the potential to raise without being stuck at a local maximum potential Nash equilibrium. Note that $\mathbf{s}^5$ is also a strong Nash equilibrium.■

In the next section, we will confine our attention to *basic* additive social interactions games $BADG$ and prove our main result on the existence of a strong landscape equilibrium in the $BADG$ class.

5 Existence of Strong Landscape Equilibria in the $BADG$ class

In order for $\mathbb{F}(\mathbf{s})$ to be a potential function for coalitional deviations, we need to restrict the set of admissible coalitional deviations. The following lemma proves the desired result for a coalitional deviation that forms a single cycle.

Lemma 1. Suppose that game $\Gamma \in BADG$. Consider a coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}})$ which forms a single cycle $\{k_1, k_2, ..., k_q, k_{q+1}\}$ with $k_{q+1} = k_1$ in its graph representation $(M, \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}}))$. Then, we have $\mathbb{F}(\mathbf{s})$ as a weighted potential function for such a class of coalitional deviations $S(\mathbf{s}, \tilde{\mathbf{s}})$.

Proof. Suppose that a coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}})$ which forms a single $\{k_1, ..., k_q, ..., k_Q, k_{Q+1}\}$ with $k_{Q+1} = k_1$. Then, $\{k_1, ..., k_q, ..., k_Q, k_{Q+1}\} = S(\mathbf{s}, \tilde{\mathbf{s}})$, and for all $k_q \in S(\mathbf{s}, \tilde{\mathbf{s}})$, $G_q(\tilde{\mathbf{s}}) = G_q(\mathbf{s}) + S_{q-1,q}(\mathbf{s}, \tilde{\mathbf{s}}) - S_{q,q+1}(\mathbf{s}, \tilde{\mathbf{s}})$, where G_q and $S_{q,q+1}$ are abbreviations of

G_{k_q} and $S_{k_q k_{q+1}}$. Then, we have

$$\begin{aligned}\mathbb{F}(\tilde{\mathbf{s}})-\mathbb{F}(\mathbf{s}) &= \sum_{q=1}^{Q}\sum_{i\in S_{q-1,q}(\mathbf{s},\tilde{\mathbf{s}})}\sigma^i v_k^i-\sum_{q=1}^{Q}\sum_{S_{q,q+1}(\mathbf{s},\tilde{\mathbf{s}})}\sigma^i v_k^i\\ &\quad+\sum_{q=1}^{Q}\sum_{i\in S_{q-1,q}(\mathbf{s},\tilde{\mathbf{s}})}\sum_{j\in G_k(\mathbf{s})\backslash S_{q,q+1}(\mathbf{s},\tilde{\mathbf{s}})}\sigma^i p^{ij}\sigma^j\\ &\quad-\sum_{q=1}^{Q}\sum_{i\in S_{q,q+1}(\mathbf{s},\tilde{\mathbf{s}})}\sum_{j\in G_k(\mathbf{s})\backslash S_{q,q+1}(\mathbf{s},\tilde{\mathbf{s}})}\sigma^i p^{ij}\sigma^j,\end{aligned}$$

since the benefits or losses the group remaining at action k_q $(G_q(\mathbf{s})\backslash S_{q,q+1}(\mathbf{s},\tilde{\mathbf{s}}))$ receives is exactly the same as the benefits or losses the relevant departing group $S_{q,q+1}(\mathbf{s},\tilde{\mathbf{s}})$ and arriving group $S_{q-1,q}(\mathbf{s},\tilde{\mathbf{s}})$ obtaining from action k_q. Rewriting this, we obtain

$$\begin{aligned}\mathbb{F}(\tilde{\mathbf{s}})-\mathbb{F}(\mathbf{s}) &= \sum_{q=1}^{Q}\sum_{i\in S_{q-1,q}(\mathbf{s},\tilde{\mathbf{s}})}\sigma^i u^i(\tilde{\mathbf{s}})-\sum_{q=1}^{Q}\sum_{S_{q,q+1}(\mathbf{s},\tilde{\mathbf{s}})}\sigma^i u^i(\mathbf{s})\\ &= \sum_{q=1}^{Q}\sum_{i\in S_{q-1,q}(\mathbf{s},\tilde{\mathbf{s}})}\sigma^i\left(u^i(\tilde{\mathbf{s}})-u^i(\mathbf{s})\right).\end{aligned}$$

Thus, if all $i\in S(\mathbf{s},\tilde{\mathbf{s}})$ is improving by joining the coalition $S(\mathbf{s},\tilde{\mathbf{s}})$, then $\mathbb{F}(\tilde{\mathbf{s}})>\mathbb{F}(\mathbf{s})$ must hold.■

Note that the statement of Lemma 1 holds for coalitional deviation $S(\mathbf{s},\tilde{\mathbf{s}})$ which forms a path $\{k_1,k_2,...,k_q\}$ in its graph representation $(M,\mathcal{G}(\mathbf{s},\tilde{\mathbf{s}}))$ (since it can be seen as a trivial cycle), by simply setting $S_{Q,Q+1}(\mathbf{s},\tilde{\mathbf{s}})(=S_{Q,1}(\mathbf{s},\tilde{\mathbf{s}}))=\emptyset$.

The next lemma follows from Lemma 1.

Lemma 2. Suppose that game $\Gamma\in BADG$. Consider a profitable Type 2 coalitional deviation $S(\mathbf{s},\tilde{\mathbf{s}})$. Then, there is a subset $T\subseteq S(\mathbf{s},\tilde{\mathbf{s}})$ such that $T=S(\mathbf{s},(\tilde{\mathbf{s}}_T,\mathbf{s}_{-T}))$ is a profitable coalitional deviation that forms a cycle (or a path).

Proof. Since $S(\mathbf{s},\tilde{\mathbf{s}})$ is a landscape coalitional deviation, any component of $\mathcal{G}(\mathbf{s},\tilde{\mathbf{s}})$ is composed of a cycle, an out-tree, or a combination of cycles and out-trees in a directed graph. Suppose that there is a terminating node h: i.e., $S_{kh}(\mathbf{s},\tilde{\mathbf{s}})\neq\emptyset$, but $S_{h\ell}(\mathbf{s},\tilde{\mathbf{s}})=\emptyset$ for all $\ell\neq h$. Then, $S_{kh}(\mathbf{s},\tilde{\mathbf{s}})$ itself is a profitable coalitional deviation since Γ is local. Thus, $\{k,h\}$ forms a trivial cycle (a path). Suppose that there is no terminal node. Then, $\mathcal{G}(\mathbf{s},\tilde{\mathbf{s}})$ is composed of a cycle, or non-overlapping cycles. In the latter case, pick a cycle. Such a cycle forms a profitable coalitional deviation, since Γ is local. We have completed the proof.■

Lemmas 1 and 2 yield our main existence result:

Theorem 2. Every game $\Gamma \in BADG$ admits a strong landscape equilibrium. In particular, a weighted-potential-maximizing strategy profile is a strong landscape equilibrium.

Proof of Theorem 2. Let $\bar{\mathbf{s}} \in \arg\max_{\mathbf{s}} \mathbb{F}(\mathbf{s})$. Suppose that $\bar{\mathbf{s}}$ is not immune to a Type 2 coalitional deviation $S(\bar{\mathbf{s}}, \tilde{\mathbf{s}}) \neq \emptyset$. By Lemma 2, there is a profitable coalitional deviation that forms a cycle or a path. By Lemma 1, there is a strategy profile $\tilde{\mathbf{s}}$ with $\mathbb{F}(\tilde{\mathbf{s}}) > \mathbb{F}(\bar{\mathbf{s}})$, a contradiction.□

In the next section we analyze the implications of the neutrality assumption.

6 Strong landscape Equilibria in the $NBADG$ Class

The importance of the neutrality assumption is reinforced by Example 4 below, that in absence of neutrality, a potential-maximizing strategy profile is not-necessarily Pareto-efficient. Thus, we focus the analysis in this section to the $NBADG$ class.

First, note that Theorem 2 generalizes Le Breton et al. (2021) and Dower et al. (2024) who considered the games in the $NBADS$ class that allow various types of coalitional deviations. Their results continue to hold without *neutral* preferences over actions.

Corollary 1 (Le Breton et al., 2021). For every game in $BADG$, a weighted-potential-maximizing strategy profile is a landscape equilibrium.

Consider a game with two actions $|M| = 2$.[6] Then, a directed graph $(m, \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}}))$ of any coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}})$ has either a single edge out-tree or a cycle, and belongs to Type 2 class. Thus, in this special case, the sets of strong Nash equilibria and strong landscape equilibria coincide, while both are a subset of the set of landscape equilibria. Thus, we have:

Corollary 2 (Dower et al., 2024). For every game in $BADG$ with two feasible actions, a weighted-potential-maximizing strategy profile is a strong Nash equilibrium.

We now turn to a further examination of strong landscape equilibria in the $NBADG$ class. Now, consider a σ^i-weighted Benthamite social welfare:

$$\begin{aligned}\mathbb{W}(\mathbf{s}) &\equiv \sum_{k \in M} \left[\sum_{i \in G_k(\mathbf{s})} \sigma^i \left(v_k^i + \sum_{j \in G_k(\mathbf{s})} p^{ij} \sigma^j \right) \right] \\ &= \sum_{k \in M} \left[\sum_{i \in G_k(\mathbf{s})} \sigma_k^i v_k^i + \sum_{i \in G_k(\mathbf{s})} \sum_{j \in G_k(\mathbf{s})} \sigma^i p^{ij} \sigma^j \right].\end{aligned}$$

[6] Note that in the case where $p_{ij} \leq 0$ for all i and j, the potential maximization for a $NBADG$ game is equivalent to the celebrated MAXCUT problem in combinatorial optimization (Goemans and Williamson, 1995) which is known to be NP-hard.

Under neutrality, we have $v_k^i = 0$ for all $k \in M$ and $i \in N$, and $\mathbb{F}(\mathbf{s})$ becomes:

$$\mathbb{W}(\mathbf{s}) = \sum_{k \in M} \left[\sum_{i \in G_k(\mathbf{s})} \sum_{j \in G_k(\mathbf{s})} \sigma^i p^{ij} \sigma^j \right].$$

The potential $\mathbb{F}$ in this case becomes:

$$\mathbb{F}(\mathbf{s}) \equiv \frac{1}{2} \sum_{k \in M} \left[\sum_{i \in N} \sum_{j \in N} \sigma_k^i p^{ij} \sigma_k^j \right] = \frac{1}{2} \mathbb{W}(\mathbf{s}).$$

Thus, we have $\mathbb{F}(\mathbf{s}) = \frac{1}{2}\mathbb{W}(\mathbf{s})$, we have an SLE version of Le Breton et al. (2021).

Proposition 1. For every game $\Gamma \in NBADG$, a strategy profile $\mathbf{s}$ that maximizes potential function $\mathbb{F}(\mathbf{s})$ is a strongly Pareto optimal strong landscape equilibrium.

Although Proposition 1 shows that the potential-maximizing strategy profile is not only a strong landscape equilibrium but also is a strongly Pareto efficient one under neutrality, the next example shows that a potential-maximizing strategy profile may not be Pareto-efficient when neutrality is dropped. In fact, the constructed game does not admit a strongly Pareto efficient Nash equilibrium.

Example 4. Suppose that game $\Gamma \in BADG$ has four players $N = \{1, 2, 3, 4\}$ and three actions $M = \{a, b, c\}$ with the following payoff information:

v_k^i	1	2	3	4
a	2.6	2.6	-10	-10
b	-10	-10	2.6	2.6
c	0	0	0	0

$$P = \begin{pmatrix} 0 & -1 & 1.5 & 1.5 \\ -1 & 0 & 1.5 & 1.5 \\ 1.5 & 1.5 & 0 & -1 \\ 1.5 & 1.5 & -1 & 0 \end{pmatrix}$$

We set $\sigma^i = 1$ for all $i \in N$. In this example, there are four relevant strategy profiles: $\mathbf{s}^1$, $\mathbf{s}^2$, $\mathbf{s}^3$, and $\mathbf{s}^4$:

$\mathbf{s}$	(u^1, u^2, u^3, u^4)	$\mathbb{F}(\mathbf{s})$
$\mathbf{s}^1 = (a, a, b, b)$	$(1.6, 1.6, 1.6, 1.6)$	8.4
$\mathbf{s}^2 = (c, c, c, c)$	$(2, 2, 2, 2)$	4
$\mathbf{s}^3 = (a, c, c, c)$	$(2.6, 3, 0.5, 0.5)$	4.6
$\mathbf{s}^4 = (a, c, b, c)$	$(2.6, 1.5, 2.6, 1.5)$	6.7

The potential-maximizing strategy profile is $\mathbf{s}^1$, achieving $u(\mathbf{s}^1) = (1.6, 1.6, 1.6, 1.6)$ and $\mathbb{F}(\mathbf{s}^1) = 8.4$. This is a strong landscape equilibrium, Pareto dominated by $\mathbf{s}^2$, which is not even a Nash equilibrium. In fact, $\mathbf{s}^1$ is unique Nash equilibrium of this game.■

The next example shows that the Benthamite-welfare-maximizing strong landscape equilibrium is not necessarily a strong Nash equilibrium, even if the game belongs to $NBADG$ and admits a strong Nash equilibrium. This example is a *dichotomous*, where $p^{ij} \in \{-1, 1\}$, and there are only (equally desirable) allies and (equally undesirable) foes for all players. This negative result is quite robust.

Example 5. Suppose that game $\Gamma \in NBADG$ has twelve players and five feasible actions. The game is dichotomous and we set $\sigma^i = 1$ for all $i \in N$. Let propensity matrix P be:

$$P = \begin{pmatrix} \mathbf{0} & 1 & 1 & \mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 \\ 1 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ \mathbf{1} & -1 & -1 & \mathbf{0} & 1 & 1 & \mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 \\ -1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ \mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 & \mathbf{0} & 1 & 1 & \mathbf{1} & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & -1 & -1 & -1 \\ \mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 & \mathbf{0} & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

The Benthamite social welfare is maximized under $\mathbf{s}$ which generates three groups $G(\mathbf{s}) = (\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\})$ with its payoff vector $(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$, and this partition is a potential-maximizing strong landscape equilibrium ($\mathbb{F}(\mathbf{s}) = 12$). However, players $\{1, 4, 7, 10\}$ can deviate from $\mathbf{s}$ to create $\mathbf{s}'$ which generates $G(\mathbf{s}') = (\{2, 3\}, \{5, 6\}, \{8, 9\}, \{11, 12\}, \{1, 4, 7, 10\})$ with its payoff vector $(3, 1, 1, 3, 1, 1, 3, 1, 1, 3, 1, 1)$. Thus, $\mathbf{s}$ is not a strong Nash equilibrium. In contrast, $\mathbf{s}'$ is not Benthamite-welfare-maximizing ($\mathbb{F}(\mathbf{s}') = 10$), but it is a strong Nash equilibrium.■

The following example shows that there may also be a strictly Pareto-dominated strong landscape equilibrium, even though Le Breton et al. (2021) show that a potential-maximizing partition is always a strongly Pareto-efficient landscape equilibrium under neutrality.

Example 6. Suppose that game $\Gamma \in NBADG$ has six players, and six actions with the following propensity matrix P:

$$P = \begin{pmatrix} 0 & 1 & 1 & 1.5 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & -1 \\ 1.5 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$$

We set $\sigma^i = 1$ for all $i \in N$. The Benthamite social welfare is maximized under $\mathbf{s}$ which generates two groups $G(\mathbf{s}) = (\{1, 2, 3\}, \{4, 5, 6\})$ with its payoff vector $(2, 1, 1, 2, 1, 1)$, and this partition is a strong landscape equilibrium. Actually, $\mathbf{s}$ is a strong Nash equilibrium. However, $G(\mathbf{s}') = (\{1,4\},\{2\},\{3\},\{5\},\{6\})$ with its payoff vector $(1.5, 0, 0, 1.5, 0, 0)$ is also a strong landscape equilibrium.■

Finally, we show that there may be multiple strong Nash equilibria, while only one of them is potential-maximizing. In other words, there may be a strong Nash equilibrium that is not potential-maximizing even under neutrality.

Example 7. Suppose that game $\Gamma \in NBADG$ with six players, and six actions in a neutral basic game with propensity matrix P:

$$P = \begin{pmatrix} 0 & 2.5 & 1 & -10 & 1 & 2 \\ 2.5 & 0 & 2 & 1 & -10 & -10 \\ 1 & 2 & 0 & 2 & -10 & -10 \\ -10 & 1 & 2 & 0 & 2.5 & 1 \\ 1 & -10 & -10 & 2.5 & 0 & 2 \\ 2 & -10 & -10 & 1 & 2 & 0 \end{pmatrix}$$

We set $\sigma^i = 1$ for all $i \in N$. There are two strong Nash equilibrium group structures under $\mathbf{s}$ and $\tilde{\mathbf{s}}$, which generate $G(\mathbf{s}) = (\{1,2,3\},\{4,5,6\})$ with its payoff vector $(3.5, 4.5, 3, 3.5, 4.5, 3)$, and $G(\tilde{\mathbf{s}}) = (\{2,3,4\},\{5,6,1\})$ with its payoff vector $(3, 3, 4, 3, 3, 4)$, respectively. The former is a potential-maximizer, but the latter is not.■

7 Strong Landscape Equilibria in Local ADG Class with an Outside Option

In this section we consider the possibility of having an outside option: players can choose their action from an augmented action set $\bar{M} \equiv M \cup \{0\}$. We regard an outside option 0 as a special action with a $n \times n$ propensity matrix $P_{00} \equiv (\mathbf{0})_{i,j \in N}$, all of whose elements are zero. We naturally set $v_0^i = 0$ for all $i \in N$. We call a local ADG game with an outside option M**-uniform** if $P_{kk} = P$ for all $k \in M$, and $P_{00} = (\mathbf{0})_{i,j \in N}$. Note that such a game does not necessarily satisfy uniformity and Theorem 2 may not be applicable in this domain. Moreover, the following simple example demonstrates a possibility of nonexistence of a strong landscape equilibrium in presence of an outside option.

Example 8. Suppose that game $\Gamma \in ADG$ with three players $N = \{1,2,3\}$ and a single action $M = \{a\}$ and an outside option 0 available for all players. The propensity matrix is given by

$$P = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix},$$

and $v_a^1 = -0.5$, $v_a^2 = v_a^3 = -2.5$. Set $\sigma^i = 1$ for all $i \in N$. We set $\sigma^i = 1$ for all $i \in N$.

The payoff of each player i is determined by

$$u_k^i(\mathbf{s}) = v_k^i + \sum_{j \in N} p^{ij}$$

for all $i \in N$ and $k \in M$, while $u_0^i = 0$ for all $i \in N$ irrespective of who are choosing 0. There are five relevant strategy profiles:

$\mathbf{s}$	(u^1, u^2, u^3)
$\mathbf{s}^1 = (0, 0, 0)$	$(0, 0, 0)$
$\mathbf{s}^2 = (0, a, a)$	$(0, 0.5, 0.5)$
$\mathbf{s}^3 = (a, a, a)$	$(0.5, -0.5, 2.5)$
$\mathbf{s}^4 = (a, 0, a)$	$(1.5, 0, -0.5)$
$\mathbf{s}^5 = (a, 0, 0)$	$(-0.5, 0, 0)$

As is clear from the table, $\mathbf{s}^1$ is a unique Nash equilibrium. But it is not immune to a joint deviation by players 2 and 3. Thus, there is no strong landscape equilibrium in this game.■

There are two factors that contributed to this negative result: Non-neutral preferences and negative externalities generated by a player joining the existing group. To remove the first obstacle, we impose the neutrality of preferences: i.e., $v_k^i = 0$ for all $k \in M$ and all $i \in N$, and $u_0^i(G_0) = 0$ for any $i \in G_0 \subseteq N$.[7]

Proposition 2. For every neutral, local and M-uniform game in ADG with an outside option 0, a potential-maximizing strategy profile over $\bar{M} = M \cup \{0\}$ constitutes a strong landscape equilibrium.

Proof. Let $\mathbf{s}^* \in \arg\max_{\mathbf{s}} \mathbb{F}(\mathbf{s})$, where

$$\begin{aligned} \mathbb{F}(\mathbf{s}) &\equiv \frac{1}{2} \sum_{k \in M} \sum_{i \in G_k(\mathbf{s})} \sum_{j \in G_k(\mathbf{s})} \sigma^i p^{ij} \sigma^j + \frac{1}{2} \sum_{i \in G_0(\mathbf{s})} 0 \\ &= \frac{1}{2} \sum_{k \in M} \sum_{i \in G_k(\mathbf{s})} \sum_{j \in G_k(\mathbf{s})} \sigma^i p^{ij} \sigma^j. \end{aligned}$$

Note that by Theorem 2, there is no profitable Type 2 coalitional deviation $(S, \tilde{\mathbf{s}})$ with (i) $s^{i*} \neq 0$ and (ii) $\tilde{s}^i \neq 0$ for all $i \in S$. Thus, if there is a profitable deviation $(S, \tilde{\mathbf{s}})$, there has to be a player who violates either (i) or (ii). Suppose that there is $i \in S$ with $s^{i*} = k$ and $\tilde{s}^i = 0$. This implies that player i had a negative payoff under $\mathbf{s}^*$: $\sum_{j \in G_k(\mathbf{s})} \sigma^i p^{ij} \sigma^j < 0$. This implies that all members of $G_k(\mathbf{s})$ other than i together benefit $-\sum_{j \in G_k(\mathbf{s})} \sigma^i p^{ij} \sigma^j > 0$ by player i's departure to the outside option 0. Thus, $\mathbb{P}(\mathbf{s}^{-i*}, 0) > \mathbb{F}(\mathbf{s}^*)$ holds, which is a contradiction. Now, suppose that there

[7]Proposition 2 offers an extension of Dower et al. (2024) to the case, where in addition to the Estern and Western modes of development, each Aftrican country can select an independent non-alignment path.

is a group of players S with $s^{i*} = 0$ and $\tilde{s}^i = k$ for all $i \in S$ with $u^i(\tilde{\mathbf{s}}) > 0$. Since $\mathbf{s}^* \in \arg\max_{\mathbf{s}} \mathbb{F}(\mathbf{s})$, $\mathbb{F}(\tilde{\mathbf{s}}) < \mathbb{F}(\mathbf{s}^*)$. Thus, noting $u^i(\mathbf{s}^*) = 0$ for all $i \in S$, we have

$$\begin{aligned} \mathbb{F}(\tilde{\mathbf{s}}) - \mathbb{F}(\mathbf{s}^*) &= \sum_{i \in G_k(\tilde{\mathbf{s}})} \sum_{j \in G_k(\tilde{\mathbf{s}})} \sigma^i p^{ij} \sigma^j - \sum_{i \in G_k(\mathbf{s}^*)} \sum_{j \in G_k(\mathbf{s}^*)} \sigma^i p^{ij} \sigma^j \\ &= \sum_{i \in S} \sum_{j \in S} \sigma^i p^{ij} \sigma^j + 2 \times \frac{1}{2} \sum_{i \in S} \sum_{j \in G_k(\mathbf{s}^*)} \sigma^i p^{ij} \sigma^j \\ &= \sum_{i \in S} \sigma^i \left(\sum_{j \in S} p^{ij} \sigma^j + \sum_{j \in G_k(\mathbf{s}^*)} p^{ij} \sigma^j \right) = \sum_{i \in S} \sigma^i u^i(\tilde{\mathbf{s}}) > 0, \end{aligned}$$

a contradiction. Hence, neither case can happen, which completes the proof.□

The next example show, that in contrast to Corollary 2, even if there are only two feasible actions ($|M| = 2$), the potential-maximizing strategy profile is not, in general, a strong Nash equilibrium with an outside option.

Example 9. Suppose that game $\Gamma \in NADG$ with eight players and two actions $M = \{a, b\}$ and an outside option 0, is local and M-uniform, and has the following propensity matrix P:

$$P = \begin{pmatrix} 0 & -10 & 3 & 2.5 & -10 & -10 & 0.5 & 0.5 \\ -10 & 0 & 2.5 & 3 & 0.5 & 0.5 & -10 & -10 \\ 3 & 2.5 & 0 & -10 & -1 & -1 & -10 & -10 \\ 2.5 & 3 & -10 & 0 & -10 & -10 & -1 & -1 \\ -10 & 0.5 & -1 & -10 & 0 & 2 & -10 & -10 \\ -10 & 0.5 & -1 & -10 & 2 & 0 & -10 & -10 \\ 0.5 & -10 & -10 & -1 & -10 & -10 & 0 & 2 \\ 0.5 & -10 & -10 & -1 & -10 & -10 & 2 & 0 \end{pmatrix}$$

We set $\sigma^i = 1$ for all $i \in N$. In this game, players 1 and 2 will never be in the same group. Similarly, the same statement applies for players 3 and 4. Players 5 and 6 cannot be in the same group with players 1, 4, 7, and 8. The same argument applies for players 7 and 8 with regard to players 2, 3, 5, and 6.. Three relevant strategy profiles are (since the game is neutral except the outside option, a and b are interchangeable):

	G_a	G_b	G_0
$\mathbf{s}^*$	$\{1, 3\}$	$\{2, 4\}$	$\{5, 6, 7, 8\}$
$\mathbf{s}'$	$\{2, 3, 5, 6\}$	$\{1, 4, 7, 8\}$	$\emptyset$
$\mathbf{s}''$	$\{2, 5, 6\}$	$\{1, 7, 8\}$	$\{3, 4\}$

The resulting payoffs and the values of the potential are:

	1	2	3	4	5, 6	7, 8	$\mathbb{F}$
$\mathbf{s}^*$	3	3	3	3	0	0	6
$\mathbf{s}'$	3.5	3.5	0.5	0.5	1	1	5
$\mathbf{s}''$	1	1	0	0	3	3	4

Note that all members of the group $S(\mathbf{s}^*, \mathbf{s}') = \{1, 2, 5, 6, 7, 8\}$ are strictly better off by this deviation, while $\mathbb{F}(\mathbf{s}^*) > \mathbb{F}(\mathbf{s}')$ holds. Thus, the potential-maximizing strategy profile is not a strong Nash equilibrium.$\blacksquare$

To remove the second obstacle to the existence of a strong landscape equilibrium outlined in Example 7, we consider the case with non-negative externalities only, where $p^{ij} \geq 0$ holds for any $i, j \in N$. Although the logic of the proof does not use our functional specification, we show the existence of strong Nash equilibrium without neutrality in the case of $|M| = 2$.

Proposition 3. Suppose that $p^{ij} \geq 0$ holds for any $i, j \in N$ in a local and M-uniform ADG with an outside option. If $|M| = 2$, there is a strong Nash equilibrium, and, generically, there are at most two strong Nash equilibria.

Proof. In this case, even if $p^{ij} = 0$, there is no negative impact from other players joining an existing group. Thus, this case can be considered as a (weakly) positive externalities case. With an outside option, maximizing $\mathbb{F}(\mathbf{s})$ may not be useful, since the resulting action is not necessarily individually rational. In what follows, it is more convenient to use group structures $\mathbf{G} \equiv (G_a, G_b, G_0)$ instead of $\mathbf{s}$, where $G_k \subseteq N$ denotes the set of players at action $k \in \{a, b, 0\}$. We consider the following algorithm.

1. Let $\mathbf{G}^0 \equiv (G_a^0, G_b^0, G_0^0) = (\emptyset, N, \emptyset)$.
2. Suppose that $\mathbf{G}^{t'} \equiv (G_a^{t'}, G_b^{t'}, G_0^{t'})$ has been defined for all $t' = 0, ..., t$. Let $\mathbf{G}^{t+1} \equiv (G_a^{t+1}, G_b^{t+1}, G_0^{t+1})$ with $G_b^{t+1} \equiv \{i \in G_b^t : u_b^i(G_b^t) > 0\}$ and $G_0^{t+1} \equiv \{i \in G_0^t : u^i(G_b^t) \leq 0\} \cup G_0^t$.

Due to positive externalities, $G_b^{t+1} \subseteq G_b^t$ for all $t = 0, 1, ...$, and there is T such that $\mathbf{G}^t = \mathbf{G}^T$ for all $t \geq T$ due to finiteness of N. Let $\bar{\mathbf{G}} = \mathbf{G}^T$. If $\bar{\mathbf{G}}$ is a strong Nash equilibrium, we are done. Thus, suppose that $\bar{\mathbf{G}}$ is not a strong Nash equilibrium. Then, a subgroup of $S \subseteq \bar{G}_0 \cup \bar{G}_b$ wants to join a together. Let $S_b = S \cap \bar{G}_b$ and $S_0 = S \cap \bar{G}_0$. Let S move to a. That is, there is $\mathbf{G}' = (G'_a, G'_b, G'_0)$ where $G'_a = \bar{G}_a \cup S_b \cup S_0$, $G'_b = \bar{G}_b \backslash S_b$, and $G'_0 = \bar{G}_0 \backslash S_0$ such that $u^i(\mathbf{G}') > u^i(\bar{\mathbf{G}})$ for all $i \in S$. Let $\bar{\mathbf{G}}' = (\bar{G}'_a, \bar{G}'_b, \bar{G}'_0)$ with $\bar{G}'_b \equiv \{i \in G'_b : u_b^i(G'_b) > 0\}$ and $\bar{G}'_0 \equiv \left\{i \in \bar{G}_b : u_b^i(\bar{G}_b) \leq 0\right\} \cup \bar{G}_0$. If $\bar{\mathbf{G}}'$ is a strong Nash equilibrium, we are done. Otherwise, there are $S' \subset \bar{G}'_b \cup \bar{G}'_0$ and $\mathbf{G}'' = (\bar{G}''_a, \bar{G}''_b, \bar{G}''_0)$, where $G''_a = \bar{G}'_a \cup S'_b \cup S'_0$, $G''_b = \bar{G}'_b \backslash S'_b$, and $G''_0 = \bar{G}'_0 \backslash S'_0$ such that $u^i(\mathbf{G}'') > u^i(\bar{\mathbf{G}}')$ for all $i \in S'$. Let $\bar{\mathbf{G}}'' = (\bar{G}''_a, \bar{G}''_b, \bar{G}''_0)$ with $\bar{G}''_b \equiv \{i \in G''_b : u_b^i(G''_b) > 0\}$ and $\bar{G}''_0 \equiv \left\{i \in \bar{G}'_0 : u_b^i(G''_b) \leq 0\right\} \cup \bar{G}'_0$. In this procedure, $\bar{G}_a \subseteq \bar{G}'_a \subseteq \bar{G}''_a \subseteq ...$, and the members' payoff u_a^i does not decline. In contrast, $\bar{G}_b \supseteq \bar{G}'_b \supseteq \bar{G}''_b \supseteq ...$, and the members' payoff u_b^i does not rise. If we continue the process until there is no further deviations, we arrive at a strong Nash equilibrium. Similarly, we can launch our algorithm with $\mathbf{G}^0 \equiv (G_a^0, G_b^0, G_0^0) = (N, \emptyset, \emptyset)$, instead. The same procedure converges to a strong Nash equilibrium. Thus, generically, there are at most two strong Nash equilibria.$\square$

A corollary of the above result relates to games with a single action.

Corollary 3. Suppose that $p^{ij} \geq 0$ holds for any $i, j \in N$ in a local and M-uniform ADG with an outside option. If $M = \{a\}$ and $p^{ij} \geq 0$ holds for any $i, j \in N$, there is a strong Nash equilibrium. Naturally, there are multiple Nash equilibria where a group of players chooses a, whereas the union of those groups constitutes a strong Nash equilibrium.

8 Nonlinear Population Externalities

Here, we introduce nonlinear population externalities on the size of groups in the $BADG$ class. Let $n_k \in \mathbb{Z}_+$ be the population of action k, and let $\varphi_k(n_k)$ be *anonymous* population interactions at action k. We do not place any condition on population interactions function $\varphi_k : \mathbb{Z}_+ \to \mathbb{R}$ except for $\varphi_k(0) = 0$ for all k. One interpretation of φ_k function is that of a (negative) congestion cost function. Another is a finite analog of the Beckmann (1957) who considered a finite set of locations M on a line, one of which is to selected by each player. Players interact with all other players, producing knowledge from these bilateral interactions. The bilateral knowledge production is discounted by the distance between each pair of players. Assuming homogeneous players, Beckmann (1957) analyzed patterns of population agglomerations over the line.

Formally, each player i's payoff function from $u^i\left(G_k(\mathbf{s})\right)$ is given by

$$u^i(G_k(\mathbf{s})) = v^i_k + \sum_{j \in G_k(\mathbf{s})} \sigma^j p^{ij} + \varphi_k(|G_k(\mathbf{s})|). \tag{3}$$

In the case of uniform influence, i.e., $\sigma^i = 1$ for all $i \in N$, Theorem 1 can be extended to include the population externality.

Proposition 4 (Le Breton and Weber, 2011). Suppose that we have $\sigma^i = 1$ for all $i \in N$. Then, there is a Nash equilibrium for an ADG with population externalities.

We have a few remarks here:

Remark 1. Chakrabarti et al. (2025) prove the existence of a Nash equilibrium under a more general structure of the last term in (4). They considere a general externality term $H^i(\mathbf{s})$ instead of $\varphi_k(|G_k(\mathbf{s})|)$, and investigate under what conditions on $H^i(\mathbf{s})$, the existence of a Nash equilibrium is preserved. Proposition 4 is a special case of their result.

Remark 2. If there is no dyadic social interaction term in the payoff function, Proposition 4 represents an extension of Proposition 4.1 of Konishi et al. (1997a). When in addition anonymous population externalities are negative, Konishi et al. (1997b) show that a game without dyadic social interactions possesses a strong Nash equilibrium without assumptions on the functional form of the payoff functions. In fact, the study of games without the interaction term goes back to Rosenthal (1973) who considered a congestion game and showed that there exists a Nash equilibrium in pure strategies when congestion is *anonymous*—players care about their route choice and the number of commuters who use the same route. See also Holzman and Law-Yone (1997), Milchtaich (1996), Quint and Shubik (1994) for further results on the existence of Nash and strong Nash equilibria in group formation games with *anonymity* and either positive or negative externalities.

Remark 3. The above game can be extended further by introducing a pair of locations (residential and work) as each player's action, along with commuting costs. Konishi and Osawa (2025) consider a commuter-worker location choice model in an urban landscape with both production- and residential-externalities (see Fujita and Ogawa 1982; Tabuchi 1986; Akamatsu and Osawa, 2020).

Now we show that with nonlinear population externalities, a $BADG$ in general does not admit a strong landscape equilibrium.

Example 10. Suppose that game $\Gamma \in BADG$ with three players $N = \{1, 2, 3\}$, three actions $M = \{a, b, c\}$, with $\sigma^i = 1$ for all $i \in N$, and $\varphi_k(1) = \varphi_k(2) = 0$ and $\varphi_k(3) = 1.5$ for all $k \in M$. Players' preference information is summarized by $v^1_a = v^2_a = 0.2$, $v^1_b = v^2_b = -10$, $v^1_c = v^2_c = 0$, $v^3_a = -10$, $v^3_b = 0$, $v^3_c = 0.3$, and

$$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

In this example, players 1 and 2 choose either actions a or c, and player 3 chooses either actions b or c. Thus, the following actions are feasible:

1. $G_a = \{1, 2\}$, $G_b = \emptyset$, $G_c = \{3\}$: $(u^1, u^2, u^3) = (1.2, 1.2, 0.3)$
2. $G_a = \emptyset$, $G_b = \emptyset$, $G_c = \{1, 2, 3\}$: $(u^1, u^2, u^3) = (1.5, 1.5, -0.2)$
3. $G_a = \emptyset$, $G_b = \{3\}$, $G_c = \{1, 2\}$: $(u^1, u^2, u^3) = (1, 1, 0)$
4. $G_a = \{1, 2\}$, $G_b = \{3\}$, $G_c = \emptyset$: $(u^1, u^2, u^3) = (1.2, 1.2, 0)$
5. $G_a = \{1\}$, $G_b = \{3\}$, $G_c = \{2\}$: $(u^1, u^2, u^3) = (0.2, 0, 0)$
6. $G_a = \{1\}$, $G_b = \emptyset$, $G_c = \{2, 3\}$: $(u^1, u^2, u^3) = (0.2, -1, -0.7)$

Strategy profiles 1 and 3 are the only Nash equilibria, but they are not strong landscape equilibria, since the pair $\{1, 2\}$ can conduct a profitable coalitional deviation.■

The existence of a strong landscape equilibrium can however be shown by extending Theorem 2 for the case where the population externalities are weighted by the influence parameters σ^js. The externalities can be positive or negative, but they must be same for all actions $k \in M$.

Proposition 5. Consider the game with population externalities that are written by $\tilde{\varphi}(G_k) = \alpha \sum_{j \in G_k} \sigma^j$ where α is a real number (negative or positive). Then, the game can be described as a game $G \in BADG$, and G admits a strong landscape equilibrium.

Proof. Let $\hat{p}^{ij} \equiv p^{ij} + \alpha$ for all $i, j \in N$, and let $\hat{P}$ be a propensity matrix with $\hat{p}^{ij}$s. Then, since $\varphi(G_k) = \alpha \sum_{j \in G_k} \sigma^j$, $u^i(\mathbf{s}) = v^i_k + \sum_{j \in G_k(\mathbf{s})} p^{ij}\sigma^j + \alpha \sum_{j \in G_k(\mathbf{s})} \sigma^j$ holds for $s^i = k$, thus we have

$$\begin{aligned}\sigma^i u^i(\mathbf{s}) &= \sigma^i v^i_k + \sigma^i \sum_{j \in G_k(\mathbf{s})} p^{ij}\sigma^j + \sigma^i \alpha \sum_{j \in G_k(\mathbf{s})} \sigma^j \\ &= \sigma^i v^i_k + \sum_{j \in G_k(\mathbf{s})} \sigma^i \hat{p}^{ij} \sigma^j.\end{aligned}$$

Using the same potential function $\mathbb{F}$ and replacing p^{ij} with $\hat{p}^{ij}$, we have a modified strong potential function:

$$\mathbb{F}(\mathbf{s}) \equiv \sum_{k \in M} \sum_{i \in G_k(\mathbf{s})} \sigma^i v^i_k + \frac{1}{2} \sum_{k \in M} \sum_{i \in G_k(\mathbf{s})} \sum_{j \in G_k(\mathbf{s})} \sigma^i \hat{p}^{ij} \sigma^j.$$

The rest of the proof is the same as the one of Theorem 2.□

9 Re-scaling Payoffs

We have demonstrated that any game $BADG$ class is a weighted potential game. If the propensity matrix is not symmetric, could it be the case that the game is part of the BADG class once the payoffs of each player i are transformed by a monotone affine transformation f^i. In the two player case, a simple sign symmetry condition is sufficient.

Proposition 6. Suppose that $|N| = 2$ with payoffs: $\tilde{u}^i(\mathbf{s}) = \tilde{v}^i_k + w^{ij}$ if $s^i = s^j = k$, and $\tilde{u}^i(\mathbf{s}) = \tilde{v}^i_k$ if $s^i = k \neq s^j$ for all $k \in M$, and all $i, j \in N$ with $i \neq j$. If either $w^{12} \times w^{21} > 0$ or $w^{12} = w^{21} = 0$, then affine transformations of the payoff functions exist such that the new game is in the $BADG$ class.

Proof. If $w^{12} = w^{21} = 0$, let $u^i(\mathbf{s}) = \tilde{u}^i(\mathbf{s})$, and we are done. If $w^{12} \times w^{21} > 0$, let $u^i(\mathbf{s}) \equiv \frac{1}{|w^{ij}|}\tilde{u}^i(\mathbf{s}) = \frac{\tilde{v}^i_k}{|w^{ij}|} + \frac{w^{ij}}{|w^{ij}|}$. Let $p^{ij} = \frac{w^{ij}}{|w^{ij}|}$. Then, p^{ij} is either 1 or -1. Since $w^{12} \times w^{21} > 0$, $p^{ij} = p^{ji}$ holds, which completes the proof.□

However, in the game with more than two players, the following example shows that the situation is more complicated,

Example 11. Suppose that game Γ has three players $|N| = 3$ with neutral quasi-linear utility function $\tilde{u}^i = \sum_{j \in G_k} w^{ij}$, where w^{ij} denotes player i's benefit from player j in the same group G_k. We set $\sigma^i = 1$ for all $i \in N$. Now, suppose further that $0 < w^{13} < w^{12}$, $0 < w^{21} < w^{23}$, and $0 < w^{32} < w^{31}$. Then, there is no symmetric propensity matrix P. If these three players' preferences are respected, by symmetry, we have $0 < p^{13} < p^{12} = p^{21} < p^{23} = p^{32} < p^{31} = p^{13}$. It is impossible to satisfy these inequalities, similarly to the nonexistence of stable matching in a roommate problem. This example illustrates the necessity of a common ranking over all pairs of players consistent with players' preferences (see the *common-ranking property* in Banerjee et al. 2001).$\blacksquare$

Example 11 refers to preferences over single partner players, but there may be multiple partners in a group in our additive social interactions game. Finding a common ranking over all subsets of players is harder than the issue described above. A single ranking over all subsets of players is not necessarily consistent, even if we could find a single ranking over all pairs of players. This issue becomes more complicated as the number of players increases. It is hard to identify an additively separable preference profile that has a weighted potential function $\mathbb{F}(\mathbf{s})$.

Profiles that maximize a weighted potential of a game of the $BADG$ class occupy a central place in this paper. If a weighted potential game admits several weighted potentials, then as stated by Monderer and Shapley (1996), the potential maximizers remain the same. In contrast to the affine case, if the new game with transformed payoff functions is an ordinal potential game, the set of potential maximizers in the transformed game may be different if all f^i are simply monotone transformations.

Example 12. Suppose that game $\Gamma \in NBADG$ has six players $N = \{1, 2, 3, 4, 5, 6\}$ and six neutral actions $M = \{a, b, c, d, e, f\}$ with a propensity matrix:

$$P = \begin{pmatrix} 0 & -10 & 1.8 & -10 & 0.6 & 2 \\ -10 & 0 & 2 & 0.6 & -10 & 0.4 \\ 1.8 & 2 & 0 & -10 & 0.4 & -10 \\ -10 & 0.6 & -10 & 0 & 2 & 1.8 \\ 0.6 & -10 & 0.4 & 2 & 0 & -10 \\ 2 & 0.4 & -10 & 1.8 & -10 & 0 \end{pmatrix}$$

We set $\sigma^i = 1$ for all $i \in N$. Under this P, each agent's preferences over relevant coalitions are:

$$\begin{array}{rcl} \boxed{\{1,3,5\}} & \succ & _1\{\mathbf{1},\mathbf{6}\} \succ_1 \{1,3\} \succ_1 \{1,5\} \succ_1 \{1\} \\ \{\mathbf{2},\mathbf{3}\} & \succ & _2\boxed{\{2,4,6\}} \succ_2 \{2,4\} \succ_2 \{2,6\} \succ_2 \{2\} \\ \boxed{\{1,3,5\}} & \succ & _3\{\mathbf{2},\mathbf{3}\} \succ_3 \{1,3\} \succ_3 \{3,5\} \succ_3 \{3\} \\ \boxed{\{2,4,6\}} & \succ & _4\{\mathbf{4},\mathbf{5}\} \succ_4 \{4,6\} \succ_4 \{2,4\} \succ_4 \{4\} \\ \{\mathbf{4},\mathbf{5}\} & \succ & _5\boxed{\{1,3,5\}} \succ_5 \{1,5\} \succ_5 \{3,5\} \succ_5 \{5\} \\ \boxed{\{2,4,6\}} & \succ & _6\{\mathbf{1},\mathbf{6}\} \succ_6 \{4,6\} \succ_6 \{2,6\} \succ_6 \{6\} \end{array}$$

Thus, in this game, there are two (equivalent-class) strong Nash equilibria (thus, landscape equilibria): $\mathbf{G} \equiv \{\{1,3,5\}, \{2,4,6\}\}$ and $\mathbf{G}' \equiv \{\{1,6\}, \{2,3\}, \{4,5\}\}$. The propensity matrix P yields the following values of potential under $\mathbf{G}$ and $\mathbf{G}'$ are:

$$\mathbb{F}(\mathbf{G}) = \frac{1}{2}\mathbb{W}(\mathbf{G}) = \frac{1}{2}\{2.4 + 1 + 2.2 + 2.4 + 1 + 2.2\} = 5.6,$$

and

$$\mathbb{F}(\mathbf{G}') = \frac{1}{2}\mathbb{W}(\mathbf{G}') = \frac{1}{2}\{2+2+2+2+2+2\} = 6.$$

Thus, $\mathbf{G}'$ is the potential-maximizing strong Nash equilibrium. Now, we re-scale propensity matrix P slightly.

$$\hat{P} = \begin{pmatrix} 0 & -10 & 1.8 & -10 & 0.6 & \mathbf{1.85} \\ -10 & 0 & \mathbf{1.85} & 0.6 & -10 & 0.4 \\ 1.8 & \mathbf{1.85} & 0 & -10 & 0.4 & -10 \\ -10 & 0.6 & -10 & 0 & \mathbf{1.85} & 1.8 \\ 0.6 & -10 & 0.4 & \mathbf{1.85} & 0 & -10 \\ \mathbf{1.85} & 0.4 & -10 & 1.8 & -10 & 0 \end{pmatrix}.$$

This modification does not alter agents' preference orderings. However, we have:

$$\mathbb{F}_{\hat{P}}(\mathbf{G}) = \frac{1}{2}\{2.4+1+2.2+2.4+1+2.2\} = 5.6,$$

and

$$\mathbb{F}_{\hat{P}}(\mathbf{G}') = \frac{1}{2}\{1.85+1.85+1.85+1.85+1.85+1.85\} = 5.55.$$

This means that $\mathbf{G}$ is the weighted potential-maximizing strong Nash equilibrium under $\hat{P}$. Thus, potential-maximizing equilibrium selection is sensitive to the cardinal form of the payoff function even if players' ordinal preferences over coalitions remain the same. It is worth noting that while both functions are ordinal potentials, the second is not a weighted potential, which explains the emergence of different potential-maximizers in these two games. ■

10 Concluding Remarks

In this paper, we examine non-cooperative games with additive dyadic social interactions and focus on strategy profiles that are immune to various classes of group deviations. We show that there exists a strong landscape equilibrium in basic (local and uniform) social interactions games and investigate the link between potential-maximizing strong landscape equilibria and strong Nash equilibria in various settings. We also consider some extensions of our results.

We conclude this paper by illustrating how our potential function approach could be applied to a class of matching problems. Since we have been considering coalitional deviations, it is natural to explore applications of our approach to matching problems with a quota for each action. As long as a potential function is well-defined for admissible coalitional deviations, a profitable swapping of players' actions will increase the potential of the matching problem. The existence of quotas in fact limits feasible coalitional deviations. For example, consider a student assignment problem for sports activities, each having a capacity limit, and where students have dyadic preferences over their peers. We could then apply a potential-improving cyclical swapping of students, respecting quotas, to find a *landscape stable assignment* from an arbitrary initial assignment. Afacan et al. (2025) consider a seat assignment problem in public transportation with passengers possessing symmetric preferences for those sitting next to them. We can apply our approach to this problem as well.[8]

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